Rinocchio: SNARKs for Ring Arithmetic

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Abstract. Succinct non-interactive arguments of knowledge (SNARKs) enable non-interactive efficient verification of NP computations and admit short proofs. However, all current SNARK constructions assume that the statements to be proven can be efficiently represented as either Boolean or arithmetic circuits over finite fields. For most constructions, the choice of the prime field $\mathbb{F}_p$ is limited by the existence of groups of matching order for which secure bilinear maps exist.
In this work we overcome such restrictions and enable verifying computations over rings. We construct the first designated-verifier SNARK for statements which are represented as circuits over a broader kind of commutative rings, namely those containing big enough *exceptional sets*. Exceptional sets consist of elements such that their pairwise differences are invertible. Our contribution is threefold: We first introduce Quadratic Ring Programs (QRPs) as a characterization of NP where the arithmetic is over a ring. Second, inspired by the framework in Gennaro, Gentry, Parno and Raykova (EUROCRYPT 2013), we design SNARKs over rings in a modular way. We generalize pre-existent assumptions employed in field-restricted SNARKs to encoding schemes over rings. As our encoding notion is generic in the choice of the ring, it is amenable to different settings. Finally, we propose two applications for our SNARKs. In the first one, we instantiate our construction for the Galois Ring $GR(2^k, d)$, i.e. the degree-d Galois extension of $\mathbb{Z}_{2^k}$. This allows us to naturally prove statements about circuits over e.g. $\mathbb{Z}_{2^{64}}$, which closely matches real-life computer architectures such as standard CPUs. Our second application is verifiable computation over encrypted data, specifically for evaluations of Ring-LWE-based homomorphic encryption schemes.

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1 Introduction

Proof systems have a rich history in cryptography and theory of computation [GMW86, For87, BGG+90]. They are now a fundamental building block in numerous cryptographic constructions such as public-key encryption [NY90], signature schemes [CS97], identification schemes [FFS87], anonymous credentials [CL01], secure voting [CF85], secure multi-party computation [GMW87] and, more recently, in cryptocurrencies such as ZCash [BCG+14].

Succinct proofs and verification. Zero-knowledge proofs [GMR89] provide the ability to convince a verifier about the truth of a statement without revealing the secrets involved. Zero-knowledge proofs are known to exist for all languages in NP [GMW86].

A non-interactive zero-knowledge proof (NIZK) [BFM88] is a proof system where the prover sends only one message to the verifier, and the verifier decides whether to accept or not based on its input, the message, and any public parameters. NIZKs are usually studied in the common reference string (CRS) model, where some structured string is generated in an initial setup phase and made available to everyone to prove/verify statements. A large body of work has been devoted to the design and implementation of efficient proofs for a variety of applications. For various practical scenarios, some of the crucial parameters are the amount of interaction, the proof size and how efficient is to prove or verify statements. When it comes to optimization of communication complexity in proof systems, it has been shown that statistically-sound proofs are unlikely to allow for significant improvements in proof size. It was shown in [Wee05] that when considering proof systems for NP, statistical soundness requires the prover to communicate, roughly, as much information as the size of the witness. The search for ways to beat this bound motivated the study of *computationally sound* proofs.

When restricting ourselves to *computational soundness*, proofs can be shorter than the length of the witness [BCC88]. Computationally sound proofs are called *argument systems*. Many applications also require *succinct verification*, where the verifier is able to check a nondeterministic polynomial-time computation in time that is much shorter than the time required to run the computation given the NP witness. Succinct proofs were considered by Kilian [Kil92], whose four-message construction, based on probabilistically checkable proofs (PCP), was soon after made non-interactive by Micali [Mic94] in the random oracle model. In the plain model, non-interactivity is achieved by generating a CRS during a setup phase. There has been a series of works on constructing (zero-knowledge) Succinct Non-interactive ARguments of Knowledge (zk-SNARKs) [Gro10, Lip12, BCCT12, BCI+13, GGPR13, PHGR13, Lip13, BCTV14, Gro16], which have very short proofs that can be verified very quickly. All these constructions are based on non-falsifiable assumptions [Nao03], and the result of Gentry and Wichs [GW11] shows that in the plain model, it is unlikely that SNARGs for general NP languages exist based on falsifiable assumptions. The approaches of [GGPR13, PHGR13], which led to concretely efficient proofs were generalized in [BCI+13] under the concept of Linear PCP (LPCP). LPCPs are a form of interactive proofs where security holds under the assumption that the prover is restricted to compute only linear combinations of its inputs. These proofs can then be transformed into SNARKs by means of an extractable linear-only encryption scheme, that is, an encryption scheme where a valid new ciphertext output by the adversary is an affine combination of the encryptions that the adversary sees as input. Roughly, this "limited malleability" of the encryption scheme, will force the prover to adhere to the above restriction.

1.1 SNARKs for Computation over Rings

Despite the progress we have seen in SNARKs, all existing contructions offer efficiency benefits only for proving statements which can be efficiently represented as very particular forms of computation. The works of [GGPR13, PHGR13, DFGK14] consider statements represented as circuit computations, either as a Boolean circuit with AND, OR and NOT gates, or as an arithmetic circuit with addition and multiplication over a field. The results of [BSCGT13, BCG+13] imply that random-access machine computations can be efficiently reduced to circuit satisfiability. The compiler of [BCG+13] gives an efficient reduction from the correctness of programs to arithmetic circuit satisfiability for a prime field of suitable size. However, it is clearly interesting to consider computations over other rings, like $\mathbb{Z}_{2^{32}}$ and $\mathbb{Z}_{2^{64}}$. While this can be reduced to computation over a field, emulating ring arithmetic in terms of finite field operations incurs a significant overhead [KPS18]. Computation over these rings matches models of computation in real-life programming and in computer architectures such as over CPU words. In addition, fixed and floating-point arithmetic operations that frequently come up in real-world applications (for instance in approximate, rather than exact computations such as in Machine Learning [CCKP19]), are more naturally expressed in terms of operations over these rings. The work of LegoSNARK [CFQ19] partially mitigates the efficiency issue of being tied to a unique, particular representation of computation. They achieve their results by seeing a computation as naturally consisting of different components and proposing a modular approach that uses the SNARK best suited for each component. Composition of proof gadgets is orthogonal to our work, and by extending our construction to be commit-and prove, the broader class of rings to which we can efficiently apply our SNARK adds yet another tool for works in the spirit of LegoSNARK.

Applications. Verifiable computation (VC) allows a computationally weak client to outsource evaluation of a function to a powerful server. The client can then verify that the output returned by the server is indeed correct while performing less work than what is necessary for computing the function itself. SNARKs immediately give a VC scheme, where the server performs the computation and returns a SNARK proof together with the output. There has been significant progress in the recent years in constructing protocols and implementing systems for verifiable computation that leverage SNARKs [BCG+13, BCTV14, BFR+13, CFH+15]. As has been noted in prior works [PHGR13], the performance of existing constructions deteriorate for functionalities that have "bad" arithmetic circuit representations.

In order to use existing SNARK schemes, one would have to translate the statement to a statement about circuit satisfaction over a field. This translation is expected to incur some overhead, and it is desirable that one can prove native computation. Most computer architectures, for instance, Intel x64, support primitive data-types over rings. These architectures have specially designed hardware to support fast and efficient arithmetic operations over rings. Matrix multiplication is heavily optimized for the ring $\mathbb{Z}_{2^{64}}$, and many native implementations compute over $\mathbb{Z}_{2^{64}}$ by default. Proving ring computations could also lead to building anonymous credential schemes off of standard signature schemes like RSA.

Efficiency considerations. The core problem behind efficiently simulating arithmetic over $\mathbb{Z}_{2^k}$ in SNARKs in which the underlying field is $\mathbb{F}_p$ (for a 254-bit prime p) and $k < 0.5\lceil \log p \rceil$ is that of minimizing the amount of times one has to compute the modular reduction $x \mod 2^k$ so that correctness is preserved. This operation, which we denote as bit decomposition, can be implemented for circuits over $\mathbb{F}_p$ at the cost of $m + 1$ multiplication gates, where $m = \lceil \log(x_{max}) \rceil$ and x_{max} denotes the maximum value x might attain [KPS18], given its position on the circuit and any known bounds on the inputs. Inputs provided in zero-knowledge can, themselves, be ensured to be k-bit numbers at the cost of $k + 1$ multiplication gates. Whereas placing "reduction mod 2^k" gates in a circuit could be phrased as an optimization problem, practitioners do not find it efficiently solvable in practice and often resort to heuristics [KPS18].

In other applications, there is the somewhat converse problem that the field $\mathbb{F}_p$ is not big enough to represent values in a single circuit wire. This happens, for example, if one wants to compute zkSNARKs where the statements are related to some RSA ring [DFKP16]. As each ring element then corresponds to m "words", each of them on an independent circuit wire, multiplying ring elements requires e.g. $O(m^{1.58})$ multiplication gates, applying Karatsuba's method.

1.2 Our Contributions

Our goal is to construct a (zk)-SNARK for ring computations, thus bringing the theory of proof systems closer to practice. Along the way, we tackle new technical problems, introduce useful building blocks, such as Quadratic Ring Programs (QRPs) and secure encodings over rings. Finally, we provide two applications for our SNARKs based on the QRP characterization: Privacy-preserving verifiable computation and SNARKs over $\mathbb{Z}_{2^k}$.

Quadratic Programs over Rings. Gennaro et al. [GGPR13] introduced the notion of Quadratic Span Programs (QSP) and Quadratic Arithmetic Programs (QAP) which can be used to compactly encode computations. They show how to convert any Boolean/arithmetic circuit into a QSP/QAP.

In this spirit, we define an analogue of a Quadratic Arithmetic Program (QAP) for arithmetic circuits over rings, called Quadratic Ring Program (QRP). QRPs "naturally" characterize computation on the underlying ring, which allows us to construct a SNARK *without having to emulate the ring arithmetic inside a field*, as would be required if we were to use a QAP. Furthermore, we give an explicit way to construct QRPs for rings containing big enough *exceptional sets* [BCPS18, ACD+19, DLS20], i.e. sets of elements such that their pairwise differences are invertible. We believe the notion of a QRP could be of independent interest as a generalization of existing quadratic programs.

Designated-verifier (zk)-SNARK for ring computation. The QRP characterization allows a test for satisfiability of an arithmetic circuit over a ring. To construct a succinct proof, we follow the blueprint of [GGPR13, PHGR13], where the QRP test is performed in a probabilistic way. The setup produces a structured reference string that consists of linearly homomorphic encodings, on top of which the prover is expected to compute using the (secret) witness. Under knowledge-type assumptions that we extend to encodings over rings, we prove security of our designated-verifier SNARK. In particular, we prove our construction secure under variants of the generalized q-PDH and d-PKE assumptions extended to encodings over rings, carefully addressing the technical challenges that arise in the new ring setting. These generalized assumptions were already stated for encodings over fields by prior works as [GGPR13, GMNO18] and gained some confidence as a base to build post-quantum SNARKs. Similar to the counterpart of assumptions in the field case, where for instance, the existence of secure bilinear groups limits the choice of the finite fields, our ring assumptions are also cautiously made and assumed to be plausible when care is taken about the particular choice of ring and encoding scheme. In Appendix B we show that if an encryption scheme is assumed to be a linear-only extractable encoding, then that encoding satisfies the generalized q-PDH and q-PKE assumptions over rings. Therefore, if our assumptions turn out to not hold for a non-trivial choice of ring and encoding, that would lead to an efficient encryption scheme (the encoding) over that ring which allows for more than just linear homomorphism, potentially towards a new fully/somewhat homomorphic encryption scheme.

On more efficient constructions. We take a small detour to discuss our choice of [GGPR13, PHGR13] as our reference SNARK construction. While there has been a lot of progress since [GGPR13] with contructions that offer more properties and better efficiency, these two papers constitute a crucial milestone in the SNARK landscape, upon which other constructions have been built. As our work is the first one that builds SNARKs over general commutative rings while requiring only *black-box* access to the ring's operations, we consider generalizing the foundational work as a first step and then focus on further improving their efficiency. We hope that our work sets the stage for e.g. future SNARKs over rings with very small proofs (such as [Gro16]) or SNARKs over rings with an updatable CRS, both of which we discuss below.

The state-of-the art SNARK construction of Groth16 [Gro16] is very efficient and has a proof size of three group elements. The construction is, however, in the idealized Generic Group Model (GGM). Translating the ideas behind the construction to general rings would require idealized models over rings. We can in fact construct a SNARK along the lines of the construction of Groth16 and prove security assuming that the encoding satisfies "linear only extractability", which roughly means that the only operations that can be performed over the encodings are affine. While a similar assumption over fields is plausible for an encoding based on exponentiation in a bilinear group (in the GGM), this turns out to be a strong assumption for encodings over rings. Since we do not know of candidate instantiations, we give the Groth-16 like construction in Appendix F. Formalising a suitable idealized model for rings and exploring candidate encodings for linear-only assumptions is an interesting avenue for research.

While there has been recent progress on reducing the degree of trust in preprocessing SNARKs by constructing "updatable CRS" SNARKs [GKM⁺18, MBKM19, CHM⁺20], QAP-based SNARKs give the best concrete efficiency in terms of proof size. Our characterization of ring computation as a QRP and subsequent SNARK construction inherits the need for a trusted CRS generation. This allows us to obtain better proof sizes. Moreover, in the designated-verifier setting, a trusted CRS is more acceptable in practice, since if we do not need ZK, we can simply have the verifier run the setup and send the CRS to the prover, and reuse the CRS to prove many statements.

Privacy-preserving verifiable computation. We show how our new (zk)-SNARK can be instantiated with polynomial rings in order to obtain a Verifiable Computation (VC) scheme with input and output privacy. While verifiable computation is a well-studied area, the problem of ensuring both correctness and privacy of the computation performed by untrusted machines remains one of the main concerns in this setting. The works solving this problem are far from achieving practical efficiency.

A natural generic construction for such schemes would be to consider a straightforward combination of SNARKs and FHE, where FHE allows computation over encrypted data and a SNARK is used to verify the integrity of the results of the computation. However, such a generic construction results in a large overhead even when used with the most performant state-of-the-art SNARKs for arithmetic circuits over finite fields to prove FHE evaluations. This is due to the limitation of having to use QAP/SSP-based SNARKs for proving computations over ciphertexts which are not naturally expressed as field elements. Therefore, such solutions do not scale well when the evaluation in FHE has to be emulated by arithmetic circuits over fields, and the resulting privacy-preserving VC schemes have very poor efficiency.

1.3 Comparison with Related Work

The results of [BISW17] give constructions of a designated verifier Succinct Non-interactive ARGument (SNARG) based on vector encryption over rings under the assumption that the encryption scheme satisfies linear targeted malleability. The subsequent work in [BISW18] constructs a SNARG with quasi-optimal prover complexity. Even though these works use an encoding scheme over a ring to compile the information theoretic object, the statement to be proven is represented as Boolean/arithmetic circuit satisfiability over a field, and the computation is still over $\mathbb{F}_p$. The underlying linear PCP is essentially a QSP/QAP. Crucially, in these works the statement to be proved is an arithmetic circuit over a field, whereas our motivation is proving statements that are represented over rings like $\mathbb{Z}_{2^{64}}$ or a polynomial ring $\mathcal{R}_q = \mathbb{Z}_q[Y]/(f(Y))$ directly.

In [KPP+14], Kosba et al. generalize the notion of Quadratic Arithmetic Programs over a field $\mathbb{F}$ to that of Quadratic Polynomial Programs (QPPs), which compute circuits whose wires carry values in the ring $\mathbb{F}[X]$. These polynomial circuits, where the addition and multiplication operations are over $\mathbb{F}[X]$, are introduced with the goal of representing (multi-)sets S of elements over $\mathbb{F}$. While the construction in [KPP+14] is limited to rings of polynomials over the same fields for which SNARKs à la [PHGR13] are secure, our work allows to build SNARKs for any ring R satisfying the property that it has a large subset such that the difference of the elements in the subset are invertible. Furthermore, our definition of QRP also recovers the QPP formulation as an instantiation of the underlying ring R, which we show in Appendix C.

Privacy-Preserving Verifiable Computation. To our knowledge, there are four main works that consider privacy in the context of VC. The first one is the seminal paper of Gennaro et al. [GGP10] who introduced the notion of non-interactive verifiable computation and builds it from garbled circuits and FHE. The second work is that of Goldwasser et al. [GKP+13] shows how to use a succinct single-key functional encryption scheme in order to build a VC protocol that preserves the privacy of the inputs (but not of the outputs). Both of these solutions [GGP10, GKP+13] are, however not very satisfactory in terms of efficiency.

A third work that considered the problem of ensuring correctness of privacy-preserving computation is the one by Fiore et al. [FGP14], who proposed using a VC in order to prove that the homomorphic evaluation of FHE ciphertexts has been done correctly. [FGP14] solution is inherently bound to computations of quadratic functions because the VC scheme is instantiated using homomorphic MACs.

To overcome this, the most recent work in this area by Fiore et al. [FNP20] proposes a new protocol for verifiable computation on encrypted data that supports homomorphic computations of multiplicative depth larger than 1. Towards their VC scheme, [FNP20] build a new SNARK that can efficiently handle computations of arithmetic circuits over a quotient polynomial ring $\mathcal{R}_q = \mathbb{Z}_q[Y]/(f(Y))$ for a prime number q in which the prover's costs have a minimal dependence on the degree d of $f(Y)$. Although this seems to fit the arithmetic structure for Ring-LWE schemes, it imposes many limitations due to the restriction to rings $\mathcal{R}_q$ where q is not only a prime, but it also has to match secure and efficient pairing constructions for some underlying SNARK over $\mathbb{F}_q$. Another significant impact on the performance present in the work of [FNP20] is on the prover effort to evaluate the circuit C over ciphertexts. In their VC scheme, a prover performs the homomorphic evaluation of the Ring-LWE HE without reduction modulo $f(Y)$, where $f(Y)$ is the quotient polynomial that defines $\mathcal{R}_q = \mathbb{Z}_q[Y]/(f(Y))$. Instead, it computes the circuit C over $\mathbb{Z}_q[Y]$, processing polynomials of high degrees, namely the initial degree d grows linearly with the multiplicative depth of the circuit. This has a significant overhead that adds to the delegated task itself.

We take a step further and propose a better VC scheme with privacy that follows the same blueprint: combining homomorphic encryption and a SNARK. The latter is, in turn, based on encoding schemes that take as input ciphertexts of a Ring-LWE-based HE. The use of our generic SNARK for computation over rings allows for better choices of group order q which improves over the approach in [FNP20]. Moreover, in our construction, the prover is not asked to come up with a different witness than the one obtained via the delegation task it completed. Our SNARK allows then for speed up through classical efficiency optimisations in $\mathcal{R}_q$ such as Number-Theoretic Transform (NTT). Also, we provide tools to enable the application of more advanced noise reduction techniques for the Ring-LWE scheme such as modulo switching. Furthermore, our scheme is in the plain model, while [FNP20] requires a random oracle. We give a detailed comparison of our application to privacy-preserving VC with the scheme of [FNP20] in Section 7.3.

In a recent work of [BCFK20], the authors propose a new solution to verifiable computation on encrypted data. Like [FNP20], the work of [BCFK20] uses the paradigm of combining VC and HE. However, in contrast to [FNP20] that requires the HE scheme to work with very specific parameters, the solution of [BCFK20] allows a flexible choice of HE parameters. The key idea of the protocol in [BCFK20] is a new homomorphic hash function for Galois rings. While we treat verifiable computation on encrypted data in our work too, we note that our work is more general – we construct (zk)SNARKs for ring computations. We achieve privacy-preserving verifiable computation as an application of our SNARK over suitable rings. In addition, the instantiation given in [BCFK20] uses the GKR protocol that admits class of log-space uniform circuits. Our QRP abstraction yields SNARKs for general circuit computations, albeit while making knowledge assumptions similar to analogous SNARKs for fields.

2 Preliminaries

Notation. We use κ to denote the security parameter. A function is said to be negligible if for all large enough values of the input, it is smaller than the inverse of any polynomial. We use negl to denote a negligible function. We use PPT to denote probabilistic polyonomial time machines.

If $\mathcal{A}$ is a randomized algorithm, we use $y \leftarrow \mathcal{A}(x)$ to denote that y is the output of $\mathcal{A}$ on x. We write $x \leftarrow \mathcal{X}$ to mean sampling a value x uniformly from the set $\mathcal{X}$. By writing $\mathcal{A}\|\chi_{\mathcal{A}}(\sigma)$ we denote the execution of $\mathcal{A}$ followed by the execution of $\chi_{\mathcal{A}}$ on the same input σ and with the same random coins. The output of the two are separated by a semicolon.

Whenever we talk about a ring R, unless otherwise specified, we always mean a commutative finite ring with identity. We denote the units of such a ring as R^*. Finally, we write $\mathbb{Z}_{p^k}$ to denote the ring of integers modulo p^k.

Definition 1 (SNARK). *A triple of polynomial time algorithms* $(\mathsf{Setup}, \mathsf{Prove}, \mathsf{Verify})$ *is a SNARG for an NP language $\mathcal{L}$ with corresponding relation $\mathcal{R}$, if the following properties are satisfied.*

1. *Completeness. For all $(x, w) \in \mathcal{R}$, the following holds:*

$$\Pr\left(\mathsf{Verify}(\mathsf{vk}, x, \pi) = 1 \;:\; \begin{array}{c}(\sigma, \mathsf{vk}) \leftarrow \mathsf{Setup}(1^\kappa)\\ \pi \leftarrow \mathsf{Prove}(\sigma, x, w)\end{array}\right) = 1$$

2. *Knowledge Soundness . For any PPT adversary $\mathcal{A}$, there exists a PPT algorithm $\chi_{\mathcal{A}}$ such that the following probability is negligible in κ:*

$$\Pr\left(\begin{array}{c}\mathsf{Verify}(\mathsf{vk}, \tilde{x}, \tilde{\pi}) = 1\\ \wedge\, \mathcal{R}(\tilde{x}, w') = 0\end{array} \;:\; \begin{array}{c}(\sigma, \mathsf{vk}) \leftarrow \mathsf{Setup}(1^\kappa)\\ ((\tilde{x}, \tilde{\pi}); w') \leftarrow \mathcal{A}\|\chi_{\mathcal{A}}(\sigma)\end{array}\right)$$

3. *Succinctness. For any x and w, the length of the proof π is given by $|\pi| = \mathsf{poly}(\kappa) \cdot \mathsf{polylog}(|x| + |w|)$.*

Non-black-box Extraction. The notion of *Knowledge Soundness* requires the existence of an extractor that can compute a witness whenever the adversarial prover produces a valid argument. The extractor we defined above is non-black-box and gets full access to the adversary's state, including any random coins.

Zero-Knowledge. An SNARK is zero-knowledge if it does not leak any information besides the truth of the statement.

Definition 2 (zero-knowledge Succinct Non-interactive ARgument of Knowledge (zk-SNARK)). *A zk-SNARK for a relation $\mathcal{R}$ is a SNARK for $\mathcal{R}$ with the following zero-knowledge property:*

– *Zero-knowledge. There exists a PPT simulator $(\mathcal{S}_1, \mathcal{S}_2)$ such that $\mathcal{S}_1$ outputs a simulated CRS σ and trapdoor τ; $\mathcal{S}_2$ takes as input σ, a statement x and τ, and outputs a simulated proof π; and, for all PPT adversaries $(\mathcal{A}_1, \mathcal{A}_2)$, the following is negligible in κ.*

$$\left|\Pr\left(\begin{array}{c}(x, w) \in \mathcal{R}\,\wedge\\ \mathcal{A}_2(\pi, \mathsf{state}) = 1\end{array} \;:\; \begin{array}{c}(\sigma, \mathsf{vk}) \leftarrow \mathsf{Setup}(1^\kappa)\\ (x, w, \mathsf{state}) \leftarrow \mathcal{A}_1(1^\kappa, \sigma)\\ \pi \leftarrow \mathsf{Prove}(\sigma, x, w)\end{array}\right) - \Pr\left(\begin{array}{c}(x, w) \in \mathcal{R}\,\wedge\\ \mathcal{A}_2(\pi, \mathsf{state}) = 1\end{array} \;:\; \begin{array}{c}(\sigma, \tau) \leftarrow \mathcal{S}_1(1^\kappa)\\ (x, w, \mathsf{state}) \leftarrow \mathcal{A}_1(1^\kappa, \sigma)\\ \pi \leftarrow \mathcal{S}_2(\sigma, \tau, x)\end{array}\right)\right|$$

Public vs Designated verifiability. In a publicly verifiable SNARK, there is no private verification information, i.e. $\mathsf{vk} = \emptyset$. A SNARK is designated verifiable if the proof can be verified only by a party knowing vk. Note that in the designated-verifier case, the verifier's decision bit on a proof potentially leaks some information about vk. Thus, the same common reference string cannot be reused for multiple proofs as in publicly-verifiable case. This was addressed in prior works in verifiable computation [GGP10, CKV10], by either keeping the decision bit secret from the prover, or running a fresh setup every time a proof fails verification. Note that any sound scheme can tolerate $O(\log \kappa)$ bits of leakage, and assuming that the decision bit leaks only a constant number of bits of information, one would only need to run a new setup after logarithmically-many proof rejections.

Strong Soundness. Multi-statement designated-verifier SNARKs are requiring soundness to hold even against a prover that makes adaptive queries to a proof verification oracle.

2.1 Background in Ring Theory

We now turn to recall some useful results from ring theory. Most of the results here provided are standard. While some of the known results for fields and euclidean domains (such as $\mathbb{Z}$) carry over to the more general rings we deal with, others do not. For example, one has to be careful about the fact that the rings we consider contain zero divisors, i.e. $d \in R \setminus \{0\}$ for which $\exists\, q \in R \setminus \{0\}$ such that $d \cdot q = 0$.

Lemma 1. *Let R be a finite ring. Then all non-zero elements of R are either a unit or a zero divisor.*

Proof. For every $a \in R \setminus \{0\}$, let $f_a : R \to R$ be the map given by $f_a(x) = a \cdot x$. If f_a is injective, then it has to be surjective, because R is finite. Therefore, in such a case there must exist an $x \in R$ verifying that $f_a(x) = 1$. So we conclude that a is a unit.

Assume that $f_a(x)$ is not injective. Then there exist $b, c \in R, b \neq c$, such that $a \cdot b = a \cdot c$, and thus $a \cdot (b - c) = 0$. In other words, a is a zero divisor. ■

We recall that an ideal of a ring R is an additive subgroup $I \subseteq R$ such that $r \cdot x \in I$ for any $r \in R, x \in I$. Through the paper, (x) will denote the ideal generated by $x \in R$.

Theorem 1. *Let R be a finite commutative ring with identity and let $Z(R)$ denote the set of all its zero divisors. Then the following are equivalent:*

1. *$Z(R)$ is an ideal.*
2. *$Z(R)$ is a maximal ideal.*
3. *R is local.*
4. *Every $x \in Z(R)$ is nilpotent.*

Proof. (1) $\Leftrightarrow$ (2). Assume $Z(R)$ is an ideal and it is not maximal. Then, there must exist some ideal I such that $Z(R) \subsetneq I \subsetneq R$. Which is absurd, as if $Z(R) \subsetneq I$, then I must contain a unit and hence $I = R$.

(2) $\Rightarrow$ (3). Assume R contains another maximal ideal $I \neq Z(R)$. Then either $I \subsetneq Z(R)$, in which case it is not maximal, or otherwise it contains a unit and hence $I = R$.

(3) $\Rightarrow$ (1). Let M be the maximal ideal of R. In order to see that $Z(R)$ is an ideal, let x, y be any two zero-divisors and $(x), (y)$ the ideals they generate. Because R is local, then $(x) \subset M$ and $(y) \subset M$. Since M is a proper ideal of R, then we have that $\forall r \in R$, $r \cdot (x + y) \in M$ and that $r \cdot (x + y)$ cannot be a unit. Hence, by Lemma 1, $r \cdot (x + y) \in Z(R)$.

(1) $\Rightarrow$ (4). Assume $Z(R)$ is an ideal, and assume towards contradiction some $x \in Z(R)$ such that $x^i \neq 0$ for any positive integer i. Then, as R is finite, there must exist some $i > j > 0$ such that $x^i = x^j$, from which we deduce that $x^j \cdot (x^{i-j} - 1) = 0$. As $x^j \neq 0$, then it has to be that $x^{i-j} - 1 \in Z(R)$. Then, as we also now that $x^{i-j} \in Z(R)$ and $Z(R)$ is an ideal, then $(x^{i-j} - 1) - x^{i-j} = -1 \in Z(R)$. Which is absurd, as then we would have that $Z(R) = R$.

(4) $\Rightarrow$ (1). Let $Z(R) = \{x_1, \ldots, x_m\}$. We will prove that $Z(R)$ is an ideal by showing the existence of some $z \in R$ such that $z \cdot x_j = 0$ for all $j \in [m]$, from which follows that $Z(R)$ is an ideal. We construct $z = z_m$ recursively as follows. Because x_1 is nilpotent, there exists an a_1 s.t. $x_1^{a_1+1} = 0$ but $x_1^{a_1} \neq 0$, so we define $z_1 = x_1^{a_1}$. For $i \in [m]$, we define $z_i = z_{i-1} \cdot x_i^{a_i}$, where a_i (which is possibly zero) is chosen such that $z_i \neq 0$ and $z_i \cdot x_i = 0$. Notice that a_i must exist from the fact that x_i is nilpotent. ■

Theorem 2 (Chinese Remainder Theorem). *Let $I_1, \ldots, I_m$ be m pairwise co-prime*[4] *ideals of R, i.e. $\forall i \neq j, I_i + I_j = R$. Denote $I = I_1 \cdots I_m$. Then the following map is a ring isomorphism:*

$$\begin{aligned} R/I &\to R/I_1 \times \cdots \times R/I_m \\ r \bmod I &\mapsto (r \bmod I_1, \ldots, r \bmod I_m) \end{aligned}$$

Exceptional sets. Elements which satisfy that their pairwise differences are invertible will be fundamental in our constructions. These have received different names in the literature: 'Condition (F)' sets in [BCPS18], 'exceptional sequences' in [ACD+19] and 'exceptional sets' in [DLS20]. We will stick with the latter denomination.

Definition 3. *Let $A = \{a_1, \ldots, a_n\} \subset R$. We say that A is an exceptional set if $\forall i \neq j, a_i - a_j \in R^*$. We define the Lenstra constant of R to be the size of the biggest exceptional set in R.*

We will need the following generalization of the Schwartz-Zippel lemma.

Lemma 2. *[Generalized Schwartz-Zippel Lemma [BCPS18]] Let $f : R^n \to R$ be an n-variate non-zero polynomial. Let $A \subseteq R$ be a finite exceptional set. Let $deg(f)$ denote the total degree of f. Then:*

$$\Pr_{\vec{a} \leftarrow A^n}[f(\vec{a}) = 0] \leq \frac{deg(f)}{|A|}$$

Proof. We prove by induction on the number of variables n. For $n = 1$, let $a_1 \in A$ be a root of $f(x)$. As $(x - a_1)$ is a monic polynomial, we have that $f(x) = (x - a_1)f_1(x)$, where the $deg(f_1) < deg(f)$. Any other root $a_2 \in A$ has to be a root of $f_1(x)$, as $(a_2 - a_1) \in R^*$ and $f(a_2) = 0$. Hence, we have that $f(x) = (x - a_1)(x - a_2)f_2(x)$, where the $deg(f_2) < deg(f_1)$. By iterating this argument, we conclude that $f(x)$ cannot have more roots in A than $deg(f)$ and hence $\Pr_{a \leftarrow A}[f(a) = 0] \leq (deg(f))/|A|$.

Assume now the result holds for $(n-1)$-variate polynomials. Given any n-variate polynomial $f(\vec{x}) \in R[x_1, \ldots, x_n]$, denote by $k = deg_{x_n}(f)$ the largest power of x_n appearing in any monomial of f. Then we have that:

$$f(\vec{x}) = \sum_{\ell=1}^{k} x_n^{\ell} \cdot g_{\ell}(x_1, \ldots, x_{n-1})$$

Denote by $\mathcal{E}_1$ the event $g_k(\vec{a}) = 0$. By definition of k, we know that $g_k(x_1, \ldots, x_{n-1})$ is a non-zero polynomial, so by induction hypothesis $\Pr_{\vec{a} \leftarrow A^{n-1}}[\mathcal{E}_1] \leq (deg(f) - k)/|A|$. Assuming $\neg\mathcal{E}_1$ and by applying the same reasoning as for $n = 1$, we have that $f(\vec{a}) \in R[x_n]$ has at most k roots in A, so $\Pr_{\vec{a} \leftarrow A}[f(\vec{a}) = 0 | \neg\mathcal{E}_1] \leq k/|A|$. We finalize by noting that (where the probability is taking over the choice of $\vec{a} \leftarrow A^n$):

$$\begin{aligned} \Pr[f(\vec{a}) = 0] &= \Pr[f(\vec{a}) = 0 | \neg\mathcal{E}_1] \cdot \Pr[\neg\mathcal{E}_1] + \Pr[f(\vec{a}) = 0 | \mathcal{E}_1] \cdot \Pr[\mathcal{E}_1] \\ &\leq \Pr[f(\vec{a}) = 0 | \neg\mathcal{E}_1] + \Pr[\mathcal{E}_1] \leq \frac{deg(f) - k}{|A|} + \frac{k}{|A|} \end{aligned}$$

■

[4] Such ideals are also denoted co-maximal by some authors.

Interpolation. Lagrange interpolation for sets of points $(x_i, y_i) \in R^2$ can be computed, as long as all the x_i are part of the same exceptional set $A \subset R$. This follows from either looking at the definition of Lagrange basis polynomials or, more formally, from the Chinese Remainder Theorem (Theorem 2). As an intuition of the latter approach, the ideals $(x - x_i)$ are co-prime, so there is a one-to-one correspondence between any polynomial $p(x) \in R[x]/I$, where $I = \prod_{i=1}^{d+1}(x - x_i)$, and $y_1 = p(x_1), \ldots, y_{d+1} = p(x_{d+1})$. In other words, any $p(x) \in R[x]$ of degree d is uniquely determined by its evaluation at d points of an exceptional set. For more details about the CRT argument, see e.g. [ACD$^+$19].

Galois Rings. Galois Rings are the generalization of Galois Fields to the ring case. Informally, a Galois Ring relates to integers modulo p^k in the same way a Galois Field relates to integers modulo a prime p. In the following, we provide a high level overview of their properties and arithmetic. For a more detailed introduction to Galois Rings, see [Wan03].

Definition 4. *A Galois Ring is a ring of the form $R = \mathbb{Z}_{p^k}[X]/(h(X))$, where p is a prime, k a positive integer and $h(X) \in \mathbb{Z}_{p^k}[X]$ a monic polynomial of degree $d \geq 1$ such that its reduction modulo p is an irreducible polynomial in $\mathbb{F}_p[X]$.*

Given a base ring $\mathbb{Z}_{p^k}$, there is a unique degree d Galois extension of $\mathbb{Z}_{p^k}$, which is precisely the Galois Ring provided on the previous definition. Hence, we shall denote such Galois Ring as $GR(p^k, d)$. Note that Galois Rings reconcile the study of finite fields $\mathbb{F}_{p^d} = GR(p, d)$ and finite rings of the form $\mathbb{Z}_{p^k} = GR(p^k, 1)$.

Every Galois Ring $R = GR(p^k, d)$ is a local ring and its unique maximal ideal is (p). Hence, by Theorem 1, all the zero divisors of R are furthermore nilpotent, and they constitute the maximal ideal (p). Furthermore, we have that $R/(p) \cong \mathbb{F}_{p^d}$, and thus a canonical homomorphism $\pi : R \to \mathbb{F}_{p^d}$ which can be computed by 'reducing modulo p'.

Proposition 1 ([ACD$^+$19]). *The Lenstra constant of $R = GR(p^k, d)$ is p^d.*

In this work, we will be particularly interested in Galois Rings of the form $R = GR(2^k, d)$, i.e. of characteristic 2^k, maximal ideal (2) and such that $R/(2) \cong \mathbb{F}_{2^d}$. Whenever we need to explicitly represent elements $a \in R$, we will do so as it follows from Definition 4. In that case, we will say that a is given in its *additive representation*, which consists of the residue classes

$$a \equiv a_0 + a_1 \cdot X + \ldots + a_{d-1} \cdot X^{d-1} \mod h(X), \quad a_i \in \mathbb{Z}_{2^k}. \tag{1}$$

3 Quadratic Programs over Commutative Rings

We extend Quadratic Arithmetic Programs (QAPs) from working over fields, as originally introduced in [GGPR13], to also cover commutative rings with identity. This gives us a characterization for the satisfiability of arithmetic circuits over such rings. Throughout this section, whenever we talk about rings, we restrict ourselves to commutative rings with identity.

Definition 5 (Quadratic Ring Programs (QRP)). *A Quadratic Ring Program (QRP) Q over a ring R consists of three sets of polynomials, $\mathcal{V} = \{v_k(x) : k \in [0, m]\}, \mathcal{W} = \{w_k(x) : k \in [0, m]\}, \mathcal{Y} = \{y_k(x) : k \in [0, m]\}$ and a target polynomial $t(x)$, all in $R[X]$. Let C be an arithmetic* *circuit over R with n inputs and n' outputs. We say that Q is a QRP that computes C if the following holds:*

$a_1, \ldots, a_n, a_{m-n'+1}, \ldots a_m \in R^{n+n'}$ *is a valid assignment to the input/output variables of C if and only if there exist* $a_{n+1}, \ldots, a_{m-n'} \in R^{m-n-n'}$ *such that:*

$$t(x) \text{ divides } p(x),$$

where $p(x) = V(x) \cdot W(x) - Y(x)$, $V(x) = \big(v_0(x) + \sum_{k=1}^{m} a_k \cdot v_k(x)\big)$, $W(x) = \big(w_0(x) + \sum_{k=1}^{m} a_k \cdot w_k(x)\big)$ *and* $Y(x) = \big(y_0(x) + \sum_{k=1}^{m} a_k \cdot y_k(x)\big)$.

We define the size and degree of Q to be m and $\deg(t(x))$ respectively. Given polynomials $V(x), W(x), Y(x) \in R[X]$ defined as above and corresponding to a valid assignment of the input/output wires, we will call them a QRP solution.

Let C be a circuit whose gates have fan-in two and fan-out one. To build a QRP, we will make use of an exceptional set A as follows. We will pick elements $r_g \in A$ for each multiplication gate $g \in C$ and define the target polynomial as $t(x) = \prod_{g \in C}(x - r_g)$. The $v_k(x), w_k(x)$ and $y_k(x)$ polynomials can be computed by interpolating over the same r_g's (which can be done for exceptional sets) in the same way one proceeds in the QAP case [GGPR13, PHGR13] (see our more detailed explanations below). Intuitively, QRP composition when the roots of $t(x)$ are taken from the same exceptional set A follows from the same fact polynomial interpolation does: The ideals $(x - r_g)$ are co-prime and we can thus apply the Chinese Remainder Theorem. $R[X]/(t(X)) \simeq R \times \ldots \times R$, and we have a one-to-one correspondence between $p(x) \mod t(x)$ and $p(r_1), \ldots, p(r_{deg(t(x))})$.

In Section 3.1 we prove in how to build a QRP for a multiplication sub-circuit and how to compose several QRPs into a single QRP for any arithmetic circuit, in a similar spirit to GGPR [GGPR13]. In Section 3.2 we show how to obtain the QRP directly, rather than through composition.

3.1 QRP Composition

Theorem 3. *Let C be a circuit over the ring R containing only one multiplication gate. If C has $m-1$ inputs and a single output, there is a QRP of size m and degree 1 that computes C.*

Proof. Let $t(x) = x - r$, $r \in A$, where A is the exceptional set. Define $\rho_1(X_1, \ldots, X_{m-1}) = c_0 + \sum_{i=1}^{m-1} c_i \cdot X_i$ (resp. $\rho_2(X_1, \ldots, X_{m-1}) = d_0 + \sum_{i=1}^{m-1} d_i \cdot X_i$) to be the linear polynomial corresponding to the left (resp. right) input wire of the only multiplication gate in C. For $k \in \{0, \ldots, m-1\}$, let $v_k(x) = c_k$, $w_k(x) = d_k$, and $y_k(x) = 0$. Set $v_m(x) = w_m(x) = 0$ and $y_m(x) = 1$. Then we have that:

$$\Big(v_0(x) + \sum_{k=1}^{m} a_k \cdot v_k(x)\Big) \cdot \Big(w_0(x) + \sum_{k=1}^{m} a_k \cdot w_k(x)\Big) - \Big(y_0(x) + \sum_{k=1}^{m} a_k \cdot y_k(x)\Big)$$
$$= \rho_1(a_1, \ldots, a_{m-1}) \cdot \rho_2(a_1, \ldots, a_{m-1}) - a_m = p(x)$$

We prove that this is a QRP for C. First assume that $a_1, \ldots, a_m \in R^m$ is a valid assignment to the input/output of C. Then $p(x) = 0$, which is trivially divisible by $t(x)$. Conversely, assume that the degree-zero polynomial $p(x)$ is divisible by the degree-one $t(x)$. As r is a root of $t(x)$, then so it has to be of $p(x)$, which implies $p(x) = 0$. ∎

Composing QRPs. Our definition of QRPs and the construction of QRP above, allow for their composition exactly as in the field case [GGPR13]. In the following, we use the symbol $\circ$ both for circuit and QRP composition. Note that the composition theorem below holds for the particular QRP construction of Theorem 3, and we make no claims about other constructions that satisfy the QRP definition. In particular, we are careful to pick all the roots of the target polynomials to belong to the same exceptional set A.

For $i \in \{1,2\}$, let Q_i be a QRP computing an arithmetic circuit f_i. Let $\mathcal{I}_i$ be the set of indices representing all wires in f_i and allow $\mathcal{I}_1 \cap \mathcal{I}_2$ to 'stitch' up to ℓ output wires of $\mathcal{I}_1$ to the inputs of $\mathcal{I}_2$. Denote such stitched circuit as $C = C_2 \circ C_1$. Express Q_i as $\mathcal{V}^{(i)} = \{v_k^{(i)}(x) : k \in \mathcal{I}_i\}, \mathcal{W}^{(i)} = \{w_k^{(i)}(x) : k \in \mathcal{I}_i\}, \mathcal{Y}^{(i)} = \{y_k^{(i)}(x) : k \in \mathcal{I}_i\}$ and target polynomial $t^{(i)}(x)$. Then, let $Q = Q_2 \circ Q_1$ consists of $\mathcal{V} = \{v_k(x) : k \in \mathcal{I}_1 \cup \mathcal{I}_2\}, \mathcal{W} = \{w_k(x) : k \in \mathcal{I}_1 \cup \mathcal{I}_2\}, \mathcal{Y} = \{y_k(x) : k \in \mathcal{I}_1 \cup \mathcal{I}_2\}$ and a target polynomial $t(x)$ which are constructed as follows.

First, define $t(x) = t^{(1)}(x) \cdot t^{(2)}(x)$. Second, for all indices $\tilde{k} \in \mathcal{I}_2 \setminus \mathcal{I}_1$, extend the definition of the wire polynomials in Q_1 as $v_{\tilde{k}}^{(1)}(x) = w_{\tilde{k}}^{(1)}(x) = y_{\tilde{k}}^{(1)}(x) = 0$. Proceed analogously for Q_2 and $\hat{k} \in \mathcal{I}_1 \setminus \mathcal{I}_2$. For all $k \in \mathcal{I}_1 \cup \mathcal{I}_2$ and $i \in \{1,2\}$, we can now set $v_k(x) \equiv v_k^{(i)}(x) \mod t^{(i)}(x)$, $w_k(x) \equiv w_k^{(i)}(x) \mod t^{(i)}(x)$ and $y_k(x) \equiv y_k^{(i)}(x) \mod t^{(i)}(x)$. Such modular equivalences can be satisfied as long as the target polynomials have no common roots, as we show in the following lemma.

Lemma 3. *Let $t^{(1)}(x)$, $t^{(2)}(x) \in R[X]$ be two polynomials which have roots only on the same exceptional set $A \subset R$ and such that they have no common roots. Let $I_1 = (t^{(1)}(x))$, $I_2 = (t^{(2)}(x))$ and $I = I_1 \cdot I_2$. Then $R[X]/I \xrightarrow{\sim} R[X]/I_1 \times R[X]/I_2$.*

Proof. For $i \in \{1,2\}$, let $t^{(i)}(x) = \prod_{j_i=1}^{d_i}(x - r_{j_i}^{(i)})$. Define ideals $I_{i,j_i} = (x - r_{j_i}^{(i)})$, where $1 \leq j_i \leq d_i$. Define $S = \{I_{i,j_i} : 1 \leq i \leq 2, 1 \leq j_i \leq d_i\}$. All the ideals in S are pairwise coprime. To see that, take any $K, \tilde{K} \in S$ and re-denote for simplicity $K = (x - k), \tilde{K} = (x - \tilde{k})$. As $k - x \in K$, we have that $k - \tilde{k} = k - x + x - \tilde{k} \in K + \tilde{K}$. Hence, as $k, \tilde{k}$ are two different elements from the same exceptional set $A \subset R$, we have that $k - \tilde{k}$ is a unit and so $K + \tilde{K} = R[X]$.

Given the above, we can apply the CRT (Theorem 2) three times and conclude that

$$R[X]/I_1 \times R[X]/I_2 \xrightarrow{\sim} (\prod_{j_1=1}^{d_1} R[X]/I_{1,j_1}) \times (\prod_{j_2=1}^{d_2} R[X]/I_{2,j_2}) \xrightarrow{\sim} R[X]/I.$$

■

We prove that the above construction for $Q = Q_2 \circ Q_1$ indeed computes $C = C_2 \circ C_1$.

Theorem 4. *Let C_1 and C_2 be two arithmetic circuits computed by QRPs Q_1 and Q_2. Assume the target polynomials of both QRPs have roots only on the same exceptional set $A \subset R$, but no common roots. Allow also some of the input variables of C_2 to include some ℓ output variables from C_1, but let no other kind of overlapping between the arithmetic circuits be possible. Denote by $C = C_2 \circ C_1$ the circuit obtained by stitching C_1 and C_2 together at those ℓ wires.*

There exists a QRP Q with size $|Q| = |Q_1| + |Q_2| - \ell$ and $\deg(Q) = \deg(Q_1) + \deg(Q_2)$ that computes C. Q's target polynomial is the product of the target polynomials for Q_1 and Q_2.

Proof. Let $\mathcal{I}_{i/o}, \mathcal{I}_{1,i/o}, \mathcal{I}_{2,i/o}$ be the indices of the input/output wires of C, C_1 and C_2, respectively. Suppose $\boldsymbol{a}_{i/o} = \{a_k \in \mathcal{I}_{i/o}\}$ is a valid input/output assignment for C. By definition, such input/output assignment can be extended to a valid assignment to all wires of C and hence in particular we can extend $\boldsymbol{a}_{i/o}$ to a valid assignment $\tilde{\boldsymbol{a}} = \{a_k \in \mathcal{I}_{1,i/o} \cup \mathcal{I}_{2,i/o}\}$. Since Q_1 is a QRP, there exist coefficients $\boldsymbol{b} = \{b_k : k \in \mathcal{I}_1\}$ which are consistent with the valid assignment to $\mathcal{I}_{1,i/o}$ and such that the polynomial

$$p^{(1)}(x) = \big(v_0^{(1)}(x) + \sum_{k\in\mathcal{I}_1} b_k \cdot v_k^{(1)}(x)\big) \cdot \big(w_0^{(1)}(x) + \sum_{k\in\mathcal{I}_1} b_k \cdot w_k^{(1)}(x)\big) - \big(y_0^{(1)}(x) + \sum_{k\in\mathcal{I}_1} b_k \cdot y_k^{(1)}(x)\big)$$

is a multiple of $t^{(1)}(x)$. The same reasoning can be applied to Q_2, for a polynomial $p^{(2)}(x)$ defined from coefficients $\boldsymbol{c} = \{c_k : k \in \mathcal{I}_2\}$ which must exist by the fact that Q_2 is a QRP. By construction, $\boldsymbol{b}$ and $\boldsymbol{c}$ must be consistent for the indices in $\mathcal{I}_1 \cap \mathcal{I}_2$, as those are contained in both $\mathcal{I}_{1,i/o}$ and $\mathcal{I}_{2,i/o}$, which were fixed by the extended assignment $\tilde{\boldsymbol{a}}$. Therefore, we can define $\boldsymbol{a} = \{a_k \in \mathcal{I}_1 \cup \mathcal{I}_2\}$ as $a_k = b_k$ for all $b_k \in \mathcal{I}_1$ and $a_k = c_k$ for all $c_k \in \mathcal{I}_2$. Let

$$p(x) = \big(v_0(x) + \sum_{k\in\mathcal{I}_1\cup\mathcal{I}_2} a_k \cdot v_k(x)\big) \cdot \big(w_0(x) + \sum_{k\in\mathcal{I}_1\cup\mathcal{I}_2} a_k \cdot w_k(x)\big) - \big(y_0(x) + \sum_{k\in\mathcal{I}_1\cup\mathcal{I}_2} a_k \cdot y_k(x)\big)$$

where $v_k(x), w_k(x)$ and $y_k(x)$ are defined from $v_k^{(i)}(x), w_k^{(i)}(x)$ and $y_k^{(i)}(x)$, $i \in \{1,2\}$, as described above (note the hypothesis of Lemma 3 are satisfied). We show that $t(x)$ divides $p(x)$. Since $v_k(x) = v_k^{(1)}(x) \mod t^{(1)}(x)$, $w_k(x) \equiv w_k^{(1)}(x) \mod t^{(1)}(x)$ and $y_k(x) \equiv y_k^{(1)}(x) \mod t^{(1)}(x)$ for all k, and since $v_{\tilde{k}}(x) = w_{\tilde{k}}(x) = y_{\tilde{k}}(x) \equiv 0 \mod t^{(1)}(x)$ for all $\tilde{k} \in \mathcal{I}_2 \setminus \mathcal{I}_1$, we conclude that $t^{(1)}(x)$ divides $p(x)$. Applying analogous reasoning, we can deduce that $t^{(2)}(x)$ divides $p(x)$ and, thus, $t(x) = t^{(1)}(x) \cdot t^{(2)}(x)$ divides $p(x)$

Conversely, let $p(x)$ be defined from the polynomial sets $\mathcal{V}, \mathcal{W}$ and $\mathcal{Y}$ as above and such that $t(x)$ divides $p(x)$. We show that any set of coefficients $\boldsymbol{a}$ enabling such divisibility contains a valid assignment $\boldsymbol{a}_{i/o} = \{a_k \in \mathcal{I}_{i/o}\}$ to the input/output wires of C. As $p(x) \equiv 0 \mod t(x)$, by Lemma 3, $p(x) \equiv 0 \mod t^{(i)}(x)$ for $i \in \{1,2\}$. Since Q_1 and Q_2 are QRPs, it follows that $\boldsymbol{a}$ must then contain valid assignment to the input/output wires of C_1 and C_2. As $\mathcal{I}_{i/o} \subseteq \mathcal{I}_{1,i/o} \cup \mathcal{I}_{2,i/o}$, we have found a valid assignment $\boldsymbol{a}_{i/o}$ to the input/output wires of C. ■

Finally, we conclude by showing how to build a QRP for any arithmetic circuit by using the previous results from this section.

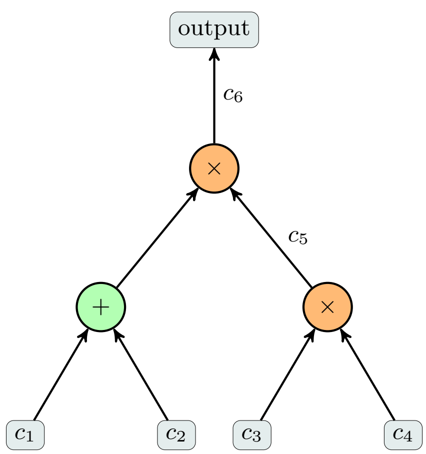
Theorem 5. *Let C be an arithmetic circuit with n inputs in (a subring of) R and $s < |A|$ multiplication gates, each with fan-in 2. If each output wire of C is the output of a multiplication gate, there is a QRP with size $n + s$ and degree s that computes C.*

Proof. We obtain this result by combining Theorem 3 and Theorem 4, one multiplication gate at a time. As long as $s < |A|$, we can ensure that the target polynomials of the QRPs for each multiplication gate do not have common roots, so that Theorem 4 can be invoked. ■

There is only one small task remaining. Let C be a circuit with $\tilde{n} \geq 1$ output wires which are not the output of multiplication gates. Our last result does not teach us how to deal with C, but we can build a modified circuit $\tilde{C}$ for which the hypothesis of Theorem 5 are satisfied. As in [GGPR13], $\tilde{C}$ has one additional 'dummy' input wire, which is required to be always assigned to the multiplicative identity 1. Furthermore, $\tilde{C}$ has a $\tilde{n}$ additional multiplication gates: For each of them, the left gate-input wire is the 'dummy' circuit-input wire and the right gate-input wire is one of the circuit-output wires which did not satisfy the hypothesis of Theorem 5. It follows that the QRP of size $n + s + \tilde{n} + 1$ and degree $s + \tilde{n}$ that computes $\tilde{C}$ also computes the original C.

3.2 Direct QRP construction.

Given a circuit C, we can construct a QRP for C using the composition theorem above. We can also construct a QRP directly for the given circuit without relying on composition. Let C be a circuit whose gates have fan-in two and fan-out one. To build a QRP, we will make use of an exceptional set A as follows. In order to define the target polynomial, we will pick elements $r_g \in A$ for each multiplication gate $g \in C$ and define $t(x) = \prod_{g \in C}(x - r_g)$. We define the polynomials $v_k(x), w_k(x)$ and $y_k(x)$ by interpolating over those same points in the same way one proceeds in the QAP case [PHGR13]. As an example for this procedure, see Figure 1.



Roots	Polynomials in QRP $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, t(x))$		
Gates	Left inputs	Right inputs	Outputs
r_5	$v_3(r_5) = 1$ $v_k(r_5) = 0,$ $k \neq 3$	$w_4(r_5) = 1$ $w_k(r_5) = 0,$ $k \neq 4$	$y_5(r_5) = 1$ $y_k(r_5) = 0,$ $k \neq 5$
r_6	$v_1(r_6) = v_2(r_6) = 1$ $v_k(r_6) = 0,$ $k \neq 1, 2$	$w_5(r_6) = 1$ $w_k(r_6) = 0,$ $k \neq 5$	$y_6(r_6) = 1$ $y_k(r_6) = 0,$ $k \neq 6$

Fig. 1. Arithmetic circuit and equivalent QRP. The polynomials $\mathcal{V} = \{v_k(x) : k \in [6]\}, \mathcal{W} = \{w_k(x) : k \in [6]\}$, $\mathcal{Y} = \{y_k(x) : k \in [6]\}$ and the target polynomial $t(x) = (x - r_5)(x - r_6)$ are defined in terms of their evaluations at two random points belonging to the same exceptional set ($r_5, r_6 \in A$), one for each multiplicative gate.

4 Secure Encoding Schemes over Rings

To construct a SNARK, we follow the framework in [GGPR13]. The QRP polynomials are represented by encodings of the polynomials evaluated at a secret point, and the encoding used is additively homomorphic in the ring of computation. We now define these encodings and their properties.

Definition 6 (Encoding scheme). *An encoding scheme* Encode *over a ring R consists of a tuple of algorithms* (Gen, E).

- $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^\kappa)$*, a key generation algorithm that takes as input a security parameter and outputs a secret key* sk*, and public information* pk.
- $s \leftarrow \mathsf{E}(a)$*, a probabilistic encoding algorithm mapping a ring element $a \in R$ to an encoding s in encoding space S such that the sets $\{\{\mathsf{E}(a)\} : a \in R\}$ partition S, where $\{\mathsf{E}(a)\}$ is the set of encodings of a. Depending on the encoding algorithm,* E *could require the secret state* sk*. To ease notation, we will omit this additional argument.*

  An encoding scheme has to satisfy the following properties:

- *ℓ-Linearly homomorphic: There is an efficient algorithm* Eval *that on input public information* pk*, encodings $\mathsf{E}(a_1), \ldots \mathsf{E}(a_\ell)$ and coefficients $c_1, \ldots, c_\ell \in R^\ell$ computes the encoding $\mathsf{E}(\sum_{i=1}^{\ell} c_i \cdot a_i)$.*
- *Quadratic root detection: There exists an algorithm that given secret key* sk*, $(\mathsf{E}(a_1), \ldots \mathsf{E}(a_d))$, and a quadratic polynomial $Q(x_1, \ldots, x_t) \in R[X_1, \ldots, X_t]$, can distinguish whether $Q(a_1, \ldots, a_t) = 0$.*
- *Image verification: There exists an efficient algorithm that given* sk*, and an element c, can detect if c is a valid encoding of some element in R.*
- *Correctness: We say that the encoding scheme is (statistically) correct if all valid encodings are decoded successfully (with overwhelming probability).*

4.1 Secure Encodings

While the definition of encoding above can be satisfied by, for instance, the identity function, we will only be interested in *secure* encodings, i.e. those which satisfy certain cryptographic assumptions. In the following, A (resp. A^*) denotes an exceptional set of a commutative ring with identity R (resp. R^*, the units of that ring).

Assumptions. We rely on computational assumptions about the encoding scheme. These have been previously used in the discrete-logarithm group setting, and here we generalize them to encodings over rings. We also show (in Appendix B) how our assumptions follow from the more intuitive, but stronger, notion of linear-only extractable encodings from [BCI+13].

We start by giving a generalized version of the q-PDH problem used in [GGPR13]. This assumption has two differences with respect to the original one. First of all, the adversary is able to win the game as long as it outputs a pair (a, y) such that $a \neq 0$ and $y \in \{\mathsf{E}(a \cdot s^{q+1})\}$. In the field case, this is trivially equivalent to the original q-PDH assumption, as $a^{-1} \cdot \mathsf{E}(a \cdot s^{q+1}) = \mathsf{E}(s^{q+1})$. Nevertheless, in the ring case, we need to deal with elements $a \in R$ which might be zero divisors. Second, in order to have the assumption work for any given q, we need to ensure that $s^{2q} \neq 0$. Due to this and additional security reasons, we restrict s to be a unit. Furthermore, we need s to be part of a big enough exceptional set, so that we can prove the soundness of our SNARKs by invoking the Generalized Schwartz-Zippel lemma.

Assumption 1 (Generalized q-PDH) *The generalized q-power Diffie-Hellman assumption holds for an encoding scheme* Encode *if, for every non-uniform* PPT *algorithm $\mathcal{A}$, the following holds:*

$$\Pr\left(a \neq 0 \wedge y \in \{\mathsf{E}(a \cdot s^{q+1})\} \; : \; \begin{array}{c} (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^\kappa), \\ s \leftarrow A^*, \\ \sigma = (\mathsf{pk}, \mathsf{E}(1), \mathsf{E}(s), \ldots, \mathsf{E}(s^q), \mathsf{E}(s^{q+2}), \mathsf{E}(s^{2q})), \\ (a, y) \leftarrow \mathcal{A}(\sigma) \end{array} \right) \leq \frac{1}{|A|} + \mathsf{negl}(\kappa).$$

Note that we linked our generalization of q-PDH to the size of the exceptional set A^*. Usually, we will consider A^* to be of exponential size in the security parameter, so that the previous probability is just negligible in the security parameter. Nevertheless, for the purpose of parallel soundness amplification techniques, in some cases it will be useful to consider even exceptional sets of constant size.

We also need a q-power knowledge assumption, which is both augmented to handle the designated verifier setting and generalized to encodings over rings.

Assumption 2 (Generalized Augmented q-PKE) *The generalized augmented q-power knowledge of encoding assumption holds for an encoding scheme* Encode *and for the class $\mathcal{Z}$ of "benign" auxiliary input generators if, for every non-uniform PPT auxiliary input generator $Z \in \mathcal{Z}$ and for all non-uniform* PPT *algorithm $\mathcal{A}$ there exists a non-uniform* PPT *extractor $\chi_{\mathcal{A}}$ such that the following probability is negligible in the security parameter:*

$$\Pr\left(\begin{array}{c} \hat{c} - \alpha c = 0 \\ \wedge \\ c \neq \sum_{i=0}^{q} a_i s^i \end{array} : \begin{array}{c} (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^\kappa), \alpha \leftarrow R^*, s \leftarrow A^*, \\ \sigma = (\mathsf{pk}, \mathsf{E}(1), \mathsf{E}(s), \ldots, \mathsf{E}(s^q), \mathsf{E}(\alpha), \mathsf{E}(\alpha s), \ldots, \mathsf{E}(\alpha s^q)), \\ z \leftarrow \mathcal{Z}(\sigma) \\ (\mathsf{E}(c), \mathsf{E}(\hat{c}); a_0, \ldots, a_q) \leftarrow (\mathcal{A}||\chi_{\mathcal{A}})(\sigma, z) \end{array} \right).$$

In the above, $(x; y) \leftarrow (\mathcal{A}||\chi_{\mathcal{A}})(\sigma, z)$ denotes that on input (σ, z), $\mathcal{A}$ outputs x, and $\chi_{\mathcal{A}}$ given the same input (σ, z), and $\mathcal{A}$'s random tape, outputs y. When we assume that $\mathcal{Z}$ is benign, we mean that the auxiliary information z is generated with a dependency on sk, s and α that is limited to the extent that it can be generated efficiently from σ.

5 Designated Verifier SNARK

The QRP characterization allows a test for satisfiability of an arithmetic circuit, by checking if the target polynomial divides $p(x) = \left(\sum c_k \cdot v_k(x)\right) \cdot \left(\sum c_k \cdot w_k(x)\right) - \left(\sum c_k \cdot y_k(x)\right)$. If divisibility holds, there is a quotient polynomial that is guaranteed to exist that serves as a witness for this test. To construct a succinct proof, we follow the blueprint of [GGPR13, PHGR13]: the QRP test is performed in a probabilistic way at a random point chosen during the setup. Toward this end, the prover is expected to give, in the proof, the polynomials $V(x), W(x), Y(x)$ computed as a linear combination of the QRP polynomials using the intermediate witness values c_k as coefficients. The prover also provides the quotient polynomial $H(x)$, and verification checks whether $V(s) \cdot W(s) - Y(s) = H(s) \cdot t(s)$, where s is the random point that is hidden from the prover. The CRS consists of an encoding of this secret point together with encodings of the QRP polynomials, and the prover homomorphically computes the elements in the proof.

Besides the security assumptions we introduced in the previous section, our designated verifier SNARK construction will mostly rely on the two following technical lemmas. The first one will be useful to define the concrete soundness error of our construction, while the second one is an analogue of [GGPR13, Lemma 10]. At a high level, this second lemma will be invoked in the security proof to ensure that, if the adversary outputs a false proof that passes verification that implicitly uses some $V(x)$ that is not in the span of the QRP polynomial set $\{v_k(x)\}$, then the reduction will be able to use that false proof to solve a q-PDH challenge.

Lemma 4. *Given an exceptional set of size n in R, we can construct another exceptional set $A = \{0, a_1, \ldots, a_{n-1} : a_i \in R^*\}$. When an exceptional set has the latter form, we say it is given in its canonical form.*

Proof. Let $B = \{b_1, \ldots, b_n\} \subset R$ be an exceptional set. For all $i \in \{1, \ldots, n-1\}$, define $a_i = b_n - b_i$. By the definition of B, we have that $a_i \in R^*$ and hence so is $(0 - a_i)$. Furthermore, $\forall i \neq j, a_i - a_j = (b_n - b_i) - (b_n - b_j) = b_i - b_j$ which is again a unit by the definition of B. ■

Lemma 5. *Let $R[x]_{\leq e}$ denote the polynomials in $R[x]$ of degree at most e. Let $R[x]^{\neg(e)}$ denote polynomials over $R[x]$ that have a zero coefficient for x^e. Let $A^* \subset R^*$ be an exceptional set. We define $A^*[x]_{\leq e}$, $A^*[x]^{\neg(e)}$ analogously. Given a set $\mathcal{U} = \{u_i(x)\} \subset R[x]_{\leq e}$ such that $|\mathcal{U}| = m$, let span($\mathcal{U}$) denote the set of polynomials that can be generated as R-linear combinations of the polynomials in $\mathcal{U}$. Let $a(x) \in A^*[x]_{\leq e+1}$ be generated uniformly at random subject to the constraint that $\{a(x) \cdot u_i(x) : u_i(x) \in \mathcal{U}\} \subset R[x]^{\neg(e+1)}$. Let $s \leftarrow A^*$. Then, if $e > m - 1$, for all algorithms $\mathcal{A}$,*

$$\Pr\left(\begin{matrix} u(x) \in R[x]_{\leq e} \wedge \\ u(x) \notin \mathit{span}(\mathcal{U}) \wedge \\ a(x) \cdot u(x) \in R[x]^{\neg(e+1)} \end{matrix} \ : \ u(x) \leftarrow \mathcal{A}(\mathcal{U}, s, a(s))\right) \leq \frac{1}{|A^*|}$$

Proof. Let $u(x) = u_0 + u_1 x + \ldots + u_e x^e \in R[x]$ and $u(x) \notin \mathit{span}(\mathcal{U})$. Define the vector $\boldsymbol{u} = (u_0, \ldots, u_e, 0)$, corresponding to the coefficients of the monomials in u and padded with a zero, and similarly define $\boldsymbol{u}_i = (u_{i,0}, \ldots, u_{i,e}, 0)$ for every $u_i(x) \in \mathcal{U}$. Then, $\boldsymbol{u}$ is not in the span of the vectors $(s^{e+1}, s^e, \ldots, 1) \bigcup_{i \in [m]} \boldsymbol{u}_i$. This follows from the assumption that $u(x) \notin \mathit{span}(\mathcal{U})$ and the fact that the last element of $\boldsymbol{u}$ is a 0 and that of $(s^{e+1}, s^e, \ldots, 1)$ is 1.

This time following the opposite order, define a vector $\boldsymbol{a} = (a_{e+1}, \ldots, a_0)$ from the coefficients of $a(x) = a_0 + \cdots + a_{e+1} \cdot x^{e+1}$. Then, $\mathcal{A}$ has the following information about $a(x)$:

$$\begin{aligned} &\langle \boldsymbol{a}, (s^{e+1}, s^e, \ldots, 1) \rangle = a(s) \\ &\langle \boldsymbol{a}, (u_{i,0}, \ldots, u_{i,e}, 0) \rangle = 0, \quad i \in [m] \end{aligned} \tag{2}$$

Where the second set of equations comes from the fact that $\{a(x) \cdot u_i(x) : u_i(x) \in \mathcal{U}\} \subset R[x]^{\neg(e+1)}$. This provides a system of $m + 1$ linear equations on the $e + 2$ coefficients $\boldsymbol{a}$, so as $e > m - 1$ by hypothesis, $\boldsymbol{a}$ appears uniformly random to $\mathcal{A}$.

Finally, assume that $\mathcal{A}$ manages to satisfy the last missing condition for $u(x)$, that is $a(x) \cdot u(x) \in R[x]^{\neg(e+1)}$, which is equivalent to $\langle \boldsymbol{a}, \boldsymbol{u} \rangle = 0$. Since $\boldsymbol{u}$ is not in the span of $(s^{e+1}, s^e, \ldots, 1) \bigcup_{i \in [m]} \boldsymbol{u}_i$, it is not a linear combination of the equations constituting the system in (2). Hence, since every $a_i \in A^*$ and $\boldsymbol{a}$ appears uniformly random to $\mathcal{A}$ (subject to the constraints provided by the system of linear equations), by looking at $\boldsymbol{u}$ as the coefficients of a polynomial $\tilde{u}(x_1, \ldots, x_{e+2}) = u_0 \cdot x_1 + u_1 \cdot x_2 + \cdots + u_e \cdot x_{e+1} + 0 \cdot x_{e+2}$, we have that $\Pr[\langle \boldsymbol{a}, \boldsymbol{u} \rangle = 0] = \Pr[\tilde{u}(\boldsymbol{a}) = 0] \leq 1/|A^*|$ as a direct consequence of the Generalized Schwartz-Zippel Lemma (Lemma 2). ■

5.1 Construction from QRP

Let C be an arithmetic circuit over R, with m wires and d multiplication gates. Let A be an exceptional set given in canonical form and $A_Q = \{0, a_1, \dots, a_{d-1}\} \subset A$. Using A_Q, define the QRP $Q = (t(x), \{v_k(x), w_k(x), y_k(x)\}_{k=0}^m)$ which computes C. Let $A^* = A \setminus A_Q$, which satisfies that $A^* \subseteq R^*$, since A is in canonical form.

We denote by $I_{io} = 1, 2, \dots \ell$ the indices corresponding to the public input and public output values of the circuit wires and by $I_{mid} = \ell + 1, \dots m$, the wire indices corresponding to non-input, non-output intermediate values. We construct a SNARK scheme $\mathsf{Rinocchio} = (\mathsf{Setup}, \mathsf{Prove}, \mathsf{Verify})$ for ring arithmetic as described in Figure 2.

$\underline{\mathsf{Setup}(1^\kappa, \mathcal{R})}$

$(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^\kappa), \quad s \leftarrow A^*, \; r_v, r_w \leftarrow R^*, \; r_y = r_v \cdot r_w$

$\alpha, \alpha_v, \alpha_w, \alpha_y \leftarrow R^*, \; \beta \leftarrow R \setminus \{0\}$

$$\begin{aligned} \mathsf{crs} = \Big(& \{\mathsf{E}(s^i)\}_{i=0}^d, \{\mathsf{E}(r_v v_k(s))\}_{k \in I_{mid}}, \{\mathsf{E}(r_w w_k(s))\}_{k \in I_{mid}}, \{\mathsf{E}(r_y y_k(s))\}_{k \in I_{mid}}, \\ & \{\mathsf{E}(\alpha r_v v_k(s))\}_{k \in I_{mid}}, \{\mathsf{E}(\alpha r_w w_k(s))\}_{k \in I_{mid}}, \{\mathsf{E}(\alpha r_y y_k(s))\}_{k \in I_{mid}}, \\ & \{\mathsf{E}(\alpha s^i)\}_{i=0}^d, \{\mathsf{E}(\beta(r_v v_k(s) + r_w w_k(s) + r_y y_k(s)))\}_{k \in I_{mid}}, \mathsf{pk} \Big) \qquad (3) \\ \mathsf{vk} = (& \mathsf{sk}, \mathsf{crs}, s, \alpha, \beta, r_v, r_w, r_y) \end{aligned}$$

$\underline{\mathsf{Prove}(\mathsf{crs}, u, w)}$

$u = (a_1, \dots, a_\ell), \; a_0 = 1,$

$w = (a_{\ell+1}, \dots, a_m)$

$v(x) = \sum_{k=0}^m a_k v_k(x)$

$v_{mid}(x) = \sum_{k \in I_{mid}} a_k v_k(x)$

$w(x) = \sum_{k=0}^m a_k w_k(x)$

$w_{mid}(x) = \sum_{k \in I_{mid}} a_k w_k(x)$

$y(x) = \sum_{k=0}^m a_k y_k(x)$

$y_{mid}(x) = \sum_{k \in I_{mid}} a_k y_k(x)$

$h(x) = \dfrac{v(x)w(x) - y(x)}{t(x)}$

$f = r_v v_{mid}(s) + r_w w_{mid}(s) + r_y y_{mid}(s)$

$A = \mathsf{E}(r_v v_{mid}(s)), \; \hat{A} = \mathsf{E}(r_v \alpha_v v_{mid}(s)),$

$B = \mathsf{E}(r_w w_{mid}(s)), \hat{B} = \mathsf{E}(r_w \alpha_w w_{mid}(s)),$

$C = \mathsf{E}(r_y y_{mid}(s)), \; \hat{C} = \mathsf{E}(r_y \alpha_y y_{mid}(s)),$

$D = \mathsf{E}(h(s)), \; \hat{D} = \mathsf{E}(\alpha h(s)), \; F = \mathsf{E}(\beta f).$

return $\pi = (A, \hat{A}, B, \hat{B}, C, \hat{C}, D, \hat{D}, F)$

$\underline{\mathsf{Verify}(\mathsf{vk}, u, \pi)}$

$\pi = (A, \hat{A}, B, \hat{B}, C, \hat{C}, D, \hat{D}, F),$

$A = \mathsf{E}(r_v V_{mid}), \quad \hat{A} = \mathsf{E}(r_v \hat{V}_{mid}),$

$B = \mathsf{E}(r_w W_{mid}), \hat{B} = \mathsf{E}(r_w \hat{W}_{mid}),$

$C = \mathsf{E}(r_y Y_{mid}), \; \hat{C} = \mathsf{E}(r_y \hat{Y}_{mid}),$

$D = \mathsf{E}(H), \; \hat{D} = \mathsf{E}(\hat{H}), \; F = \mathsf{E}(L)$

$v_{io}(x) = \sum_{i=0}^\ell a_i v_i(x)$

$w_{io}(x) = \sum_{i=0}^\ell a_i w_i(x)$

$y_{io}(x) = \sum_{i=0}^\ell a_i y_i(x)$

$L_{span} = r_v V_{mid} + r_w W_{mid} + r_y Y_{mid}$

$P = (v_{io}(s) + V_{mid}) \cdot (w_{io}(s) + W_{mid}) - (y_{io}(s) + Y_{mid})$

$$\begin{aligned} \text{Check: } & \hat{V}_{mid} = \alpha V_{mid}, \\ & \hat{W}_{mid} = \alpha W_{mid}, \\ & \hat{Y}_{mid} = \alpha Y_{mid}, \\ & \hat{H} = \alpha H \qquad (4) \\ & L = \beta L_{span} \qquad (5) \\ & P = H \cdot t(s) \qquad (6) \end{aligned}$$

Fig. 2. The Rinocchio scheme for (zk-)SNARKs over a ring R.

Zero-knowledge. We can make our construction zero-knowledge by randomizing the elements in the proof π such that the checks verify and the proof is statistically indistinguishable from random encodings. The idea is for the prover to add random multiples of $t(x)$ to the proof terms so that we can define a simulator that "fakes" the proof elements from completely random values. Specifically, the prover chooses random $\delta_v, \delta_w, \delta_y \leftarrow R$, and adds $\delta_v t(s)$ inside the encoding to $v_{mid}(s)$; $\delta_w t(s)$ to $w_{mid}(s)$; and $\delta_y t(s)$ to $y_{mid}(s)$. It is easy to see that the modified value of $p(x)$ remains divisible by $t(x)$.

The following terms should be added to crs: $\mathsf{E}(r_v t(s)), \mathsf{E}(r_w t(s)), \mathsf{E}(r_y t(s)), \mathsf{E}(\alpha_v r_v t(s)),\ \mathsf{E}(\alpha_w r_w t(s)),$ $\mathsf{E}(\alpha_y r_y t(s),\ \mathsf{E}(r_v \beta t(s)),\ \mathsf{E}(r_w \beta t(s)),\ \mathsf{E}(r_y \beta t(s))$. The prover can now compute the new values in π using the terms in the crs, and the verification proceeds as before.

Remark 1. Note that our construction has a proof size of nine elements, as opposed to eight elements in Pinocchio [PHGR13]. This saving in Pinocchio is a result of removing the repeated (with scalar α) encoding of the quotient polynomial $h(x)$. We remark that this means that in the proof of security, one cannot invoke PKE to extract the quotient polynomial. Indeed, Pinocchio does not extract the polynomial explicitly and, for the security proof to go through, they require multiplication of encoded values (multiplication in the exponent). This relies on more than just quadratic root *detection* from the encoding, it needs one quadratic *computation* in the reduction. While exponentiation admits this multiplication via pairings, we do not make the assumption that an encoding scheme in the designated verifier setting allows this in general. Therefore, we fall back on including an additional proof element, and additional CRS elements to enable this computation.

Remark 2. Another aspect of our construction that could look surprising to the reader is the definition of A^*: Why do we not include the elements in A_Q used to define the QRP? As previously discussed, we do this in order to precisely define the soundness of our construction. In some cases, as we will discuss in Section 6.3, it could be useful to use parallel repetition strategies for soundness amplification. Previous works in the field setting, using pairings, did not need to make such a concrete analysis, since if circuits are assumed to be of polynomial size in the security parameter, the probability that a randomly sampled $s \leftarrow \mathbb{F}$ would be precisely one of the points used to define the QRP would be negligible, because $\mathbb{F}$ has exponential size in the security parameter. In all rigour, nevertheless, the *concrete* soundness error of those constructions is also bounded by the size of $\mathbb{F}$ minus the size of the QRP[5]. We prefer this concrete analysis even when rings might have exceptional sets of exponential size in the security parameter.

5.2 Security proof

We are now ready to prove that $\mathsf{Rinocchio}$ satisfies the properties of a SNARK as stated in Definition 1. We remark that we do not prove *strong* soundness, which demands that soundness holds even when the prover has access to the verification oracle. While some designated-verifier schemes are provably strongly sound, the reduction requires the d-PKEQ assumption (see Assumption 3 in Appendix B) on the encoding scheme to hold. For the sake of keeping $\mathsf{Rinocchio}$ as general as possible in the choice of rings and encodings, we do not make that assumption, but our result could be adapted to that case.

Theorem 6. *Let R be commutative ring with identity with an exceptional subset A, and d be an upper bound on the degree of the QRP. Assuming that the generalized augmented $(4d+3)$-PKE and the generalized q-PDH assumptions hold for the encoding scheme* Encode *over R (and A^*) for $q = 4d+4$, the protocol* $\mathsf{Rinocchio}$ *described above is a SNARK as per Definition 1, with soundness error $1/|A^*|$.*

[5] In order to see this, consider a proof that consists purely of encodings of zero. The checks in the verification equations would pass if s happened to coincide with a value in the QRP used to describe a multiplication gate with no connections to input or output wires. This applies to e.g. [PHGR13].

Completeness. Assuming the encoding scheme Encode satisfies (statistical) correctness, then it follows by inspection that the verification equations are satisfied by a correctly generated proof π. Therefore (statistical) completeness of the Rinocchio protocol follows by QRP completeness.

Soundness. Assume there exists an adversary $\mathcal{A}$ who returns the proof of a false statement. We use this adversary $\mathcal{A}$ in order to construct an adversary $\mathcal{B}$ who breaks the q-PDH assumption.

Setting up the CRS. Adversary $\mathcal{B}$ is given the description of the encoding scheme, and the challenge $\mathsf{E}(1), \mathsf{E}(s), \ldots, \mathsf{E}(s^q), \mathsf{E}(s^{q+2}), \ldots, \mathsf{E}(s^{2q})$. $\mathcal{B}$ provides the crs to $\mathcal{A}$ by constructing it as follows. It samples $r'_v, r'_w, \alpha, \alpha_v, \alpha_w, \alpha_y$ at random from R^* and sets $r'_y = r'_v r'_w$. Let $r_v = r'_v s^{d+1}$, $r_w = r'_w s^{2(d+1)}$, and $r_y = r'_y s^{3(d+1)}$. The value β is chosen as follows. Sample a polynomial $\beta_{poly}(x) \in A^*[X]$ of degree at most $3d+3$ uniformly at random, subject to the constraint that $\beta_{poly}(x) \cdot (r'_v v_k(x) + r'_w x^{(d+1)} w_k(x) + r'_y x^{2(d+1)} y_k(x))$ has a zero coefficient for x^{3d+3} for all k. $\mathcal{B}$ sets $\beta = s^{q-(4d+3)} \beta_{poly}(s)$. Looking ahead in our proof, the polynomial $x^{q-(4d+3)} \cdot \beta_{poly}(x)$ will play the role of $a(x)$ in Lemma 5. $\mathcal{B}$ sets the CRS as follows:

$$\begin{aligned}\mathsf{crs} = \Big(&\{\mathsf{E}(s^i)\}_{i=0}^d, \{\mathsf{E}(r_v v_k(s))\}_{k \in I_{mid}}, \{\mathsf{E}(r_w w_k(s))\}_{k \in I_{mid}}, \{\mathsf{E}(r_y y_k(s))\}_{k \in I_{mid}},\\ &\{\mathsf{E}(\alpha_v r_v v_k(s))\}_{k \in I_{mid}}, \{\mathsf{E}(\alpha_w r_w w_k(s))\}_{k \in I_{mid}}, \{\mathsf{E}(\alpha_y r_y y_k(s))\}_{k \in I_{mid}},\\ &\{\mathsf{E}(\alpha s^i)\}_{i=0}^d, \{\mathsf{E}(\beta(r_v v_k(s) + r_w w_k(s) + r_y y_k(s))\}_{k \in I_{mid}}, \mathsf{pk}\Big)\end{aligned}$$

We now argue that $\mathcal{B}$ can construct the above crs using the terms provided in its challenge. Consider the term in the proof that involves β, which is the final proof term that the prover will have to compute using the CRS.

$$\begin{aligned}&\mathsf{E}(\beta(r_v v_{mid}(s) + r_w w_{mid}(s) + r_y y_{mid}(s)))\\ &= \mathsf{E}(\beta(r'_v s^{d+1} v_{mid}(s) + r'_w s^{2(d+1)} w_{mid}(s) + r'_y s^{3(d+1)} y_{mid}(s))).\end{aligned} \tag{7}$$

In this term, β is multiplied by a polynomial evaluated at s. Note that $\mathcal{B}$ generated β also as a polynomial evaluated at s. If we further rewrite (7) by expressing β in terms of s, we have

$$\begin{aligned}&\mathsf{E}(s^{q-3d-2} r'_v \beta_{poly}(s) v_{mid}(s) + s^{q-2d-1} r'_w \beta_{poly}(s) w_{mid}(s) + s^{q-d} r'_y \beta_{poly}(s) y_{mid}(s))\\ &= \mathsf{E}(s^{q-3d-2} \beta_{poly}(s)(r'_v v_{mid}(s) + s^{d+1} r'_w w_{mid}(s) + s^{2d+2} r'_y y_{mid}(s))).\end{aligned} \tag{8}$$

Since $\beta_{poly}(x) \cdot (r'_v v_k(x) + r'_w x^{d+1} w_k(x) + r'_y x^{2d+2} y_k(x))$ has a zero coefficient in front of x^{3d+3}, the value underneath the encoding in (8) has a zero in front of s^{q+1}. The powers of s in the encoding go up to $(q-3d-2)+(3d+3)+(2d+2)+d = q+3d+3 \leq 2q$. The polynomials $v_k(x), w_k(x), y_k(x)$ are of degree d, and none of the other elements in the CRS contain s^{q+1} inside the encoding. Since we have $q \geq 4d+4$, all the elements in the CRS can be generated using terms in the challenge.

We need to make sure that a crs generated as above has a distribution which is indistinguishable to the one given in our protocol. Note that, as $\beta_{poly}(x)$ is a polynomial of degree at most $3d+3$ and $\beta = s^{q-(4d+3)} \beta_{poly}(s)$, we have that $\Pr[\beta = 0] \leq (3d+3)/|A^*|$ (Lemma 2). This is a bigger chance for $\beta = 0$ than in our protocol, but notice that $\mathcal{A}$ never sees β in the clear, but rather encodings of it. There are hence two cases: Either $\mathsf{E}(0)$ is computationally indistinguishable from any $\mathsf{E}(a)$ where $a \neq 0$, or it is not (as it happens in the exponentiation-based encodings of e.g. [GGPR13, PHGR13]). In the former case, $\mathcal{A}$ will accept the crs. In the latter case, $\mathcal{B}$ checks whether $\beta = 0$ by distinguishing whether the last term of crs is $\mathsf{E}(0)$ and, if so, samples a new $\beta_{poly}(x)$ and repeats the process above until the last term is not $\mathsf{E}(0)$.

Extraction. With the CRS set this way, $\mathcal{B}$ can now obtain a purported proof from $\mathcal{A}$. Due to the indistinguishability of simulated CRS and real CRS, $\mathcal{A}$ aborting on input the tailored CRS is negligible. Let $\hat{\pi}$ be a purported proof returned by $\mathcal{A}$, which is parsed as follows:

$$\hat{\pi} = (\ \mathsf{E}(r_v V_{mid}), \mathsf{E}(r_w W_{mid}), \mathsf{E}(r_y Y_{mid}), \mathsf{E}(H),$$
$$\mathsf{E}(r_v \hat{V}_{mid}), \mathsf{E}(r_w \hat{W}_{mid}), \mathsf{E}(r_y \hat{Y}_{mid}), \mathsf{E}(\hat{H}), \mathsf{E}(L)\ \Big)$$

Since $\mathcal{B}$ knows that $r_v = r'_v s^{d+1}$, $r_w = r'_w s^{2(d+1)}$, and $r_y = r'_y s^{3(d+1)}$, it can reinterpret $\hat{\pi}$ as follows:

$$\Big(\ \mathsf{E}_{r'_v}(s^{d+1} V_{mid}), \mathsf{E}_{r'_w}(s^{2d+2} W_{mid}), \mathsf{E}_{r'_y}(s^{3d+3} Y_{mid}), \mathsf{E}(H),$$
$$\mathsf{E}_{r'_v}(s^{d+1} \hat{V}_{mid}), \mathsf{E}_{r'_w}(s^{2d+2} \hat{W}_{mid}), \mathsf{E}_{r'_y}(s^{3d+3} \hat{Y}_{mid}), \mathsf{E}(\hat{H}), \mathsf{E}(L)\ \Big)$$

Notice that the proof elements are now being treated as if they belonged to four different encodings: $\mathsf{E}, \mathsf{E}_{r'_v}, \mathsf{E}_{r'_w}, \mathsf{E}_{r'_y}$, where the four latter are defined as $\mathsf{E}_a(b) = \mathsf{E}(a \cdot b)$. It is easy to see that, by the fact that $r'_v, r'_w, r'_y \in R^*$ and the assumption that E is a secure encoding, so are $\mathsf{E}_{r'_v}, \mathsf{E}_{r'_w}, \mathsf{E}_{r'_y}$. Since $\hat{\pi}$ passes verification (in particular Equation (4)), we can apply the following reasoning for E and any of the other three encodings. As $(\mathsf{E}(H), \mathsf{E}(\hat{H}))$ is of the form $(\mathsf{E}(H), \mathsf{E}(\alpha H))$, $\mathcal{B}$ can use the d-PKE extractor χ_A to extract a polynomial $H(x) = \sum_{i=0}^{d} h_i x^i$ of degree at most d such that $H = H(s)$. This is because the CRS given to $\mathcal{A}$ is of the form (σ, z), where:

$$\sigma = (\mathsf{pk}, \{\mathsf{E}(s^i)\}_{i=0}^{d}, \{\mathsf{E}(\alpha s^i)\}_{i=0}^{d}), \quad z = \mathsf{crs} \setminus \sigma$$

Note that the auxiliary information z is independent of α, as the relation between e.g. $\mathsf{E}_{r'_v}(s^{d+1} V_{mid})$ and $\mathsf{E}_{r'_v}(s^{d+1} \hat{V}_{mid})$ is an i.i.d. α_v. If we look at any of the three remaining encodings $\mathsf{E}_{r'_v}(\cdot)$, $\mathsf{E}_{r'_v}(\cdot)$ or $\mathsf{E}_{r'_v}(\cdot)$, we will next show that $\mathcal{B}$ can extract $V_{mid}(x)$ of degree at most d and such that $V_{mid} = V_{mid}(s)$ due to the $(2d+1)$-PKE assumption (resp. $W_{mid}(x)$ due to $(3d+2)$-PKE and $Y_{mid}(x)$ due to $(4d+3)$-PKE). Focusing on $V_{mid}(x)$, notice that $\mathcal{A}$ does not have a $(2d+1)$-PKE challenge, but the following (where the problem is with $\tilde{\sigma}_v$, not with z)

$$\tilde{\sigma}_v = (\mathsf{pk}, \{\mathsf{E}_{r'_v}(s^{d+1} v_k(s))\}_{k \in I_{mid}}, \{\mathsf{E}_{r'_v}(\alpha_v s^{d+1} v_k(s))\}_{k \in I_{mid}}), \quad z = \mathsf{crs} \setminus \tilde{\sigma}_v$$

which differs from the expected $\sigma_v = (\mathsf{pk}, \{\mathsf{E}_{r'_v}(s^i)\}_{i=0}^{2d+1}, \{\mathsf{E}_{r'_v}(\alpha_v s^i\}_{i=0}^{2d+1})$ in two ways: It is completely missing the powers $\{s^i\}_{i=0}^{d}$ and, for those between $d+1$ and $2d+1$, it instead has the evaluation at s of the polynomials $\{x^{d+1} v_k(x)\}_{k \in I_{mid}}$. Informally, since $\mathcal{B}$ can compute $\tilde{\sigma}_v$ from σ_v, we can extract. In more syntactic rigour, $\mathcal{B}$ can send σ_v to a $(2d+1)$-PKE adversary $\mathcal{A}_v$ who runs internally the SNARK prover $\mathcal{A}$ on $\tilde{\sigma}_v$, so as Equation (4) verifies, then, by the PKE assumption there exists an extractor χ_{A_v} which gets a polynomial $x^{d+1} V_{mid}(x) = \sum_{i=0}^{d} v_i x^{d+1+i}$ of degree at most $2d+1$ such that $V_{mid} = V_{mid}(s)$. Applying the same reasoning, we can conclude on the extraction of polynomials $W_{mid}(x), Y_{mid}(x)$ of degree at most d such that $W_{mid} = W_{mid}(s)$ and $Y_{mid} = Y_{mid}(s)$.

Reducing to PDH. Since the proof $\hat{\pi}$ verifies but the statement is false, we show that then one of the following must hold, where $V(x) = \sum_{k \in I_{io}} c_k v_k(x) + V_{mid}(x)$ and similarly $W(x)$ and $Y(x)$:

Case 1: $V(x) \cdot W(x) - Y(x) \neq H(x) \cdot t(x)$, but Equation (6) holds, therefore, $V(s) \cdot W(s) - Y(s) = H(s) \cdot t(s)$.

Case 2: The polynomial

$$U(x) = r'_v x^{d+1} V_{mid}(x) + r'_w x^{2(d+1)} W_{mid}(x) + r'_y x^{3(d+1)} Y_{mid}(x)$$

is not in the module S generated by the R-linear combinations of the polynomials $\{u_k(x) = r'_v x^{d+1} v_k(x) + r'_w x^{2(d+1)} w_k(x) + r'_y x^{3(d+1)} y_k(x)\}_{k \in I_{mid}}$.

We demonstrate that those are the only cases for a false $\hat{\pi}$ by proving that, if none of them holds, then $V(x), W(x)$ and $Y(x)$ are a QRP solution, which would then mean that $\hat{\pi}$ is a valid proof. So, towards contradiction, assume both that $U(x) \in S$ and $V(x) \cdot W(x) - Y(x) = H(x) \cdot t(x)$. Since $U(x) \in S$, it can be expressed as $U(x) = \sum_{k \in I_{mid}} c_k u_k(x)$, where $c_k \in R$. Thus,

$$U(x) = r'_v x^{d+1} v'(x) + r'_w x^{2(d+1)} w'(x) + r'_y x^{3(d+1)} y'(x),$$

where we define $v'(x) = \sum_{k \in I_{mid}} c_k v_k(x), w'(x) = \sum_{k \in I_{mid}} c_k w_k(x)$ and $y'(x) = \sum_{k \in I_{mid}} c_k y_k(x)$. Note that $v'(x), w'(x), y'(x)$ have degree at most d, since they are in the spans of $\{v_k(x)\}_{k \in I_{mid}}, \{w_k(x)\}_{k \in I_{mid}}$ and $\{y_k(x)\}_{k \in I_{mid}}$ respectively. Since $V_{mid}(x), W_{mid}(x), Y_{mid}(x)$ are also polynomials of degree at most d, and since the R-submodules $\{x^{d+1+i} : i \in [0, d]\}$, $\{x^{2(d+1)+i} : i \in [0, d]\}$, and $\{x^{3(d+1)+i} : i \in [0, d]\}$ of $R[x]$ are disjoint (except at zero) we have that

$$\begin{aligned} U(x) &= r'_v x^{d+1} V_{mid}(x) + r'_w x^{2(d+1)} W_{mid}(x) + r'_y x^{3(d+1)} Y_{mid}(x) \\ &= r'_v x^{d+1} v'(x) + r'_w x^{2(d+1)} w'(x) + r'_y x^{3(d+1)} y'(x), \end{aligned}$$

we conclude that $V_{mid}(x) = v'(x)$, $W_{mid}(x) = w'(x)$ and $Y_{mid}(x) = y'(x)$. Therefore, $V(x) = \sum_{k \in I_{io}} c_k v_k(x) + V_{mid}(x) = \sum_{k \in I_{io}} c_k v_k(x) + \sum_{k \in I_{mid}} c_k v_k(x)$, $W(x) = \sum_{k \in I_{io}} c_k w_k(x) + W_{mid}(x) = \sum_{k \in I_{io}} c_k w_k(x) + \sum_{k \in I_{mid}} c_k w_k(x)$, and $Y(x) = \sum_{k \in I_{io}} c_k y_k(x) + Y_{mid}(x) = \sum_{k \in I_{io}} c_k y_k(x) + \sum_{k \in I_{mid}} c_k y_k(x)$. Finally, as we assumed that $V(x) \cdot W(x) - Y(x) = H(x) \cdot t(x)$, we have that $V(x), W(x), Y(x)$ can be written as the same linear combination $\{c_k\}_{k \in I_{io} \cup I_{mid}}$ of their respective sets, and that $t(x)$ divides $V(x) \cdot W(x) - Y(x)$. Therefore, $V(x), W(x), Y(x)$ are a QRP solution.

We now address the two cases corresponding to a false proof $\hat{\pi}$ and show that, in both Case 1 and Case 2, $\mathcal{B}$ can break the Generalized q-PDH (Assumption 1).

Case 1: $V(x) \cdot W(x) - Y(x) \neq H(x) \cdot t(x)$. The non-zero polynomial $\gamma(x) = V(x) \cdot W(x) - Y(x) - H(x) \cdot t(x)$ has degree $k \leq 2d$ and s as a root. Express $\gamma(x) = \gamma_k \cdot x^k + \hat{\gamma}(x)$, where $k \leq 2d$, $\gamma_k \neq 0$ and $\deg(\hat{\gamma}(x)) < k$. Since s is a root of $\gamma(x)$, it is also a root of $x^{q+1-k}\gamma(x)$. Hence, $\gamma_k \cdot s^{q+1} = -s^{q+1-k}\hat{\gamma}(s)$. $\mathcal{B}$ can compute $\mathsf{E}(\gamma_k \cdot s^{q+1})$ by computing $\mathsf{E}(-s^{q+1-k}\hat{\gamma}(s))$, which is a known linear combination of the $\{\mathsf{E}(s^i)\}_{i=0}^q$ values belonging to the q-PDH instance. This solves the q-PDH challenge.

Case 2: The polynomials $V_{mid}(x), W_{mid}(x), Y_{mid}(x)$ are not in the required spans. There does not exist $\{c_k\}_{k \in I_{mid}}$ such that $V_{mid}(x) = \sum_{k \in I_{mid}} c_k v_k(x), W_{mid}(x) = \sum_{k \in I_{mid}} c_k w_k(x)$ and $Y_{mid}(x) = \sum_{k \in I_{mid}} c_k y_k(x)$. Then, the polynomial $U(x) = r'_v x^{d+1} V_{mid}(x) + r'_w x^{2(d+1)} W_{mid}(x) + r'_y x^{3(d+1)} Y_{mid}(x)$ is not in the module S generated by the R-linear combinations of the polynomials $\{u_k(x) = r'_v x^{d+1} v_k(x) + r'_w x^{2(d+1)} w_k(x) + r'_y x^{3(d+1)} y_k(x)\}$. Recall that $\mathcal{B}$ chose a polynomial $\beta_{poly}(x) \in A^*[X]$ of degree at most $3d+3$ subject to the constraint that all polynomials in $\{\beta_{poly}(x) \cdot (r'_v v_k(x) + r'_w x^{(d+1)} w_k(x) + r'_y x^{2(d+1)} y_k(x))\}$ have a zero coefficient for x^{3d+3}. Thus, by Lemma 5, the coefficient of x^{q+1} in the polynomial $\omega(x) = x^{q-(4d+3)} \cdot \beta_{poly}(x) \cdot U(x)$ is $a \in R \backslash \{0\}$

with probability $1-1/|A^*|$. Furthermore, $\mathcal{B}$ can compute all the coefficients of $\omega(x)$ on its own, so it can subtract from $\mathsf{E}(L) = \mathsf{E}(s^{q-(4d+3)}\beta_{poly}(s)\cdot(s^{d+1}V_{mid}(s)+s^{2(d+1)}W_{mid}(s)+s^{3(d+1)}Y_{mid}(s)))$ all the monomials corresponding to $\mathsf{E}(s^j)$ for $j \neq q+1$ and obtain $\mathsf{E}(a \cdot s^{q+1})$. Note that this is possible even when $\beta_{poly}(s) = 0$. By outputting $(a, \mathsf{E}(a \cdot s^{q+1}))$, $\mathcal{B}$ breaks the generalized q-PDH assumption. ■

6 Designated Verifier SNARKs for computation over $\mathbb{Z}_{2^k}$

We instantiate our previous designated verifier SNARK using QRPs where the underlying R is the Galois Ring $GR(2^k, \delta)$. As R is a free module over $\mathbb{Z}_{2^k}$ of rank δ, we can embed elements from $\mathbb{Z}_{2^k}$ into the first coordinate of R. Hence, a QRP for arithmetic circuits over $\mathbb{Z}_{2^k}$ can be embedded in a QRP for arithmetic circuits over R. We divide this section into three smaller ones. In the first one, we discuss a suitable encoding scheme for $R = GR(2^k, \delta)$. Second, we provide a simple, direct instantiation of Theorem 6 using said encoding. Finally, we provide some QRP gadgets to perform useful computation such as bit decomposition.

6.1 A secure encoding for $GR(2^k, \delta)$

We will use the Joye-Libert (JL) cryptosystem [JL13], which has $\mathbb{Z}_{2^k}$ as message space, as building block for our encoding of Galois Ring elements.

$\mathsf{KeyGen}(1^\kappa, k)$**:** According to the security parameter κ, choose two random primes p, q satisfying the equivalences:

$$p \equiv 1 \pmod{2^k} \quad \text{and} \quad q \equiv 3 \pmod 4.$$

For simplicity, pick $p = 2^k p' + 1$ and $q = 2q' + 1$, where p', q' are primes. Let g be a random generator of both $\mathbb{Z}_p^*$ and $\mathbb{Z}_q^*$, $N = p \cdot q$ and $\mu = p'$. We define $\mathsf{pk} = (g, k, N)$ and $\mathsf{sk} = \mu$.

$\mathsf{Enc}_{\mathsf{pk}}(m)$**:** Given $m \in \mathbb{Z}_{2^k}$, sample $x \leftarrow \mathbb{Z}_N^*$ and output $C = g^m \cdot x^{2^k} \pmod N$.

$\mathsf{Dec}_{\mathsf{sk}}(C)$**:** Compute $c = C^\mu \mod p$ and then retrieve m bit by bit as follows. Observe that $c = C^\mu \mod p = (g^\mu)^m \mod p$, where g^μ is an element of order 2^k in $\mathbb{Z}_p^*$. Let $m = \sum_{j=0}^{k-1} 2^j m_j,\ m_j \in \{0,1\}$. We can compute its least significant bit m_0 by computing $c^{2^{k-1}} \mod p$. Set $m_0 = 0$ if $c^{2^{k-1}} \mod p = 1$, and 1 otherwise. After computing $m_{i-1}, \ldots, m_0$, compute m_i as follows: Set $m_i = 0$ if and only if

$$\left(\frac{c}{(g^{\mu(\sum_{j=0}^{i-1} 2^j m_j)})}\right)^{2^{k-i-1}} = 1 \mod p$$

Note that whereas the decryption cost is linear in k, there is empirical evidence [CRFG19, Section 5] that it can be faster than more common encryption schemes such as Paillier. The JL cryptosystem is secure under the assumption that k-quadratic residuosity is hard [JL13]. It is linearly homomorphic over $\mathbb{Z}_{2^k}$, and it has already been employed in the context of efficient two-party computation over $\mathbb{Z}_{2^k}$ (see [CRFG19] for concrete efficiency estimates).

Let $R = GR(2^k, \delta)$. Given $a \in R$ written in its additive form $a = a_0 + a_1 X + \ldots + a_{\delta-1} X^{\delta-1}$ (see Equation (1)), we define our encoding $\mathsf{E.JL}$ as follows:

– $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^\kappa)$ calls $\mathsf{KeyGen}(1^\kappa, k)$ in the JL cryptosystem and outputs $(\mathsf{pk}, \mathsf{sk})$.

– $\mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_\delta} \leftarrow \mathsf{E.JL}_{\mathsf{pk}}(a)$ is a probabilistic encoding algorithm mapping a ring element $a \in R$ to an encoding space $Z = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_\delta}$ such that the sets $\{\{\mathsf{E.JL}(a)\} : a \in R\}$ partition Z, where $\{\mathsf{E.JL}(a)\}$ is the set of encodings of a. Concretely:

$$\mathsf{E.JL}(a) = \big(\mathsf{Enc}_{\mathsf{pk}}(a_0), \ldots, \mathsf{Enc}_{\mathsf{pk}}(a_{d-1})\big)$$

On the suitability of the encoding. The scheme E.JL clearly satisfies all non-security properties required from an encoding. Under the security assumptions of the JL scheme, it is also reasonable to assume that q-PDH (and q-PKE) hold for the encoding scheme too, where $\mathcal{A}$ has a success probability negligible in d. This dependence on d is intrinsic to the arithmetic in R, as $\mathcal{A}$ can simply output $(a, y) = (2^{k-1}, \mathsf{E.JL}(\sum_{\ell=0}^{d-1} s_{\ell,0} \cdot X^\ell))$, where $s_{\ell,0}$ are $\mathcal{A}$'s guesses for the least significant bit of each $s_\ell \in \mathbb{Z}_{2^k}$ that conform the additive representation of s. Note that the fact that Enhanced-CPA cannot be supported by the JL cyptosystem, as brought up by [CRFG19], is not an issue here. Such notion requires interaction with an oracle which the Adversary does not have access to in our construction, as we do not aim to provide *strong* soundness.

6.2 A simple construction

We now have everything we need to instantiate the protocol defined in Section 5.1 for $R = GR(2^k, \delta)$. It follows from inspection that, representing elements of R in their additive notation, $A = \{a_i \in R : a_i = \sum_{j=0}^{\delta-1} a_{i,j} \cdot X^j, a_{i,j} \in \{0, 1\}\}$ is an exceptional set in canonical form. Let C be a circuit with d multiplication gates and define A^* as described in Section 5.1. Then, using the secure encoding scheme E.JL from Section 6.1, we can invoke Theorem 6 to obtain a DV-SNARK for $R \supset \mathbb{Z}_{2^k}$ with a soundness error of $|A^*|^{-1} = (2^\delta - d)^{-1}$.

Efficiency considerations. Whereas in this construction δ is logarithmic in the desired soundness error, we insist on the fact that our QRP does not suffer from the overhead of adding roughly k multiplication gates whenever a modular reduction $x \mod 2^k$ has to be computed, as it would happen if the circuit was to be run by a SNARK over fields (see our discussion in Section 1.1). Hence, avoiding this and using Rinocchio allows us not blow-up the *degree* of the QRP, which was an efficiency bottleneck in e.g. [PHGR13]. We would further like to note that FFT-style techniques can be applied to Galois Rings [CK91] and that the price of working with circuits over $GR(2^k, \delta)$, rather than $\mathbb{Z}_{2^k}$, has the potential to be amortized, as it has happened in the context of Multi-Party Computation protocols which faced similar limitations (c.f. [ACD+19, DLS20]).

In Appendix D we show how to build QRPs for bit decomposition, which is useful for practical bit-wise operations such as comparisons.

6.3 Soundness amplification

Despite the previous arguments, there is a concern as to what is the practical impact of the extension degree δ in the previous construction. We believe that this is an interesting question to explore in experimental work, comparing the different strategies above. We suggest one strategy here. While it would seem that we cannot escape from δ being logarithmically proportional to the soundness error, we show that it is good enough to apply a parallel amplification strategy and choose any integer $\delta > \log(d)$, where d is the number of multiplication gates in the QRP Q.

For simplicity, we will explain how to do this in the *worst* case, which is when d is a power of two. If we chose to set $\delta = \log(d) + 1$, we would then have that $|A_Q| = d$ and $|A^*| = |A| - |A_Q| = 2$.

Let our target soundness error be 2^{-S}. We can then run S independent instances of $\mathsf{Rinocchio}$ over $R = GR(2^k, \log(d) + 1)$ for the same QRP (that is, S different crs from S independent Setup executions), for each of which the prover computes the Prove step from the (common) QRP witness. The verifier only accepts if all the S proofs pass verification, yielding a soundness error of $|A^*|^{-S} = 2^{-S}$.

Let us note that this parallel repetition strategy does not greatly affect the overall proof size. The reason behind this is that working over a smaller degree extension of $\mathbb{Z}_{2^k}$ not only improves the computational efficiency, but it also reduces the size of each individual proof when using the $\mathsf{E.JL}$ encoding. In more detail, if we let N denote the ciphertext size in the Joye-Libert JL cryptosystem [JL13], our worst-case example that executes S proofs over $GR(2^k, \log(d) + 1)$ transmits exactly $9(S-1)N$ more ciphertexts than if we were to execute $\mathsf{Rinocchio}$ once over $GR(2^k, \log(d) + S)$. We would like to stress once again that this is a worst case scenario and that there is a whole trade-off space between proof size and computational efficiency to be explored, concretely $\log(d) < \delta \leq \log(d) + S$.

7 SNARKs for computation over Encrypted Data

In this section we detail how we can apply $\mathsf{Rinocchio}$ to the problem of verifiable computation over encrypted data. Our approach is generic, where we just run a proving mechanism – the (zk-)SNARK – on pre-existing Homomorphic Encryption (HE) schemes in a modular way. Taking advantage of our generic SNARK construction from Section 5, this reduces to finding secure encoding schemes over a ring that are compatible with the ciphertext space of the underlying HE scheme.

In Section 7.1 we review some popular homomorphic encryption schemes that are good candidates for realising our privacy-preserving VC scheme. Then, by using a secure encoding scheme E as the ones we provide in Section 7.2, we can invoke Theorem 6 to obtain a DV-SNARK for $\mathcal{R}_q$, as explained in Section 7.3.

7.1 Homomorphic Encryption schemes and their parameters

The first fully homomorphic encryption schemes were based on the Learning With Errors (LWE) problem [Reg05], which is the main assumption behind schemes with ciphertexts in the ring $\mathbb{Z}_q$ such as [Bra12]. Nevertheless, the most efficient HE schemes are based on the Ring-LWE problem, which is defined in [LPR10]. The LWE (resp. Ring-LWE) problem reduces, under some conditions, to hard problems on euclidean lattices (resp. ideal lattices).

In Ring-LWE-based schemes, the ring of plaintext is $\mathcal{R}_p = \mathbb{Z}_p[Y]/(f(Y))$ and the ring of cyphertexts is $\mathcal{R}_q = \mathbb{Z}_q[Y]/(f(Y))$ for some degree-N polynomial $f(Y)$. This is usually picked to be a cyclotomic polynomial, so that it factors into ℓ irreducible factors modulo p. More concretely, $f(Y) \equiv \prod_{i=1}^{\ell} f_i(Y) \mod p$, where each $f_i(Y)$ has degree $\phi(N)/\ell$. By imposing $p \equiv 1 \mod N$, this creates ℓ "plaintext slots", and hence a popular choice is $f(Y) = Y^N + 1$, where N is a power of two. In order to deal with the *noise growth* that affects all of these schemes, q is usually chosen large (several hundreds of bits). Besides the size of q, the rank of the associated lattice, which corresponds to N, has to be high enough to meet the security requirements (usually between 2^{10} and 2^{15}).

Frequently, q is chosen so that $q = \prod_{i=1}^{k} p_i$. While this does not affect the asymptotic complexity of operations on ciphertexts, it brings an important gain in practice: The polynomials of $\mathcal{R}_q$ are represented as k polynomials of same degree but with smaller coefficients, thanks to the ring isomorphism given by the CRT. Being able to efficiently deal with non-prime choices for q is hence a significant advantage of our work, compared to that of [FNP20].

Concrete Ring-LWE schemes. The works that we will consider as a candidate for HE in our privacy-preserving VC are: Brakerski and Vaikunthanatan [BV11b], BGV [BGV12], FV [FV12]. We are interested in the Somewhat Homomorphic variants of these schemes, where the parameters are set just large enough so as to enable homomorphic evaluation of some target function which will be represented as a QRP and hence fixed by the SNARK's crs.

In this setting, schemes like BGV [BGV12] use the so-called modulo-switching. They require a chain of moduli $q_0 < \cdots < q_L$ to be able to scale the noise down after each multiplication by switching the ciphertext to a smaller modulus. When evaluating circuits with large multiplicative depth, one needs to choose a large chain of moduli and thus use higher dimensions resulting in poor performance.

Scale invariant schemes allow to partially overcome this limitation by removing the need of the modulus-switching procedure which potentially results in the possibility of evaluating circuits with a bigger multiplicative depth. In his seminal work [Bra12], Brakerski introduced a new scale-invariant scheme based on classical LWE where the noise grows only linearly during multiplication removing thereby the necessity of modulus switching. This more effective noise control mechanism makes the scale-invariant schemes particularly interesting. In [FV12], the scale-invariant scheme of Brakerski is adapted to the Ring-LWE setting.

Applications for each HE scheme. From the different Ring-LWE schemes, each scheme is best suited for certain types of operations: BGV [BGV12] uses, in general, slow operations, but benefit from massively optimizations to treat many bits at the same time, while FV [FV12] allows to perform large vectorial arithmetic operations as long as the multiplicative depth of the evaluated circuit remains small. We are able to apply our SNARK for rings to proving homomorphic evaluations of these schemes by first mapping the different ciphertext spaces of the different schemes to a common algebraic structure, using some natural homomorphisms.

We observe the algebraic structure of the plaintext and ciphertext spaces from different HE schemes:

- FV: plaintexts on the ring $R_p = \mathbb{Z}_p[Y]/(f(Y))$ for some integer p (a prime, a power of 2 or a small number $1 \pmod{2N}$, depending on the functionality), ciphertexts on R_q^2
- BGV: plaintexts on the ring $R_p = \mathbb{Z}[Y]/(f(Y))$, ciphertexts on R_q^2

7.2 Secure Encodings for (Ring-)LWE ciphertexts

We introduce two different instantiations for the encoding scheme, one suitable for the ciphertext ring $\mathbb{Z}_q$ that appears in LWE-based HE and the other one for a polynomial ring $\mathcal{R}_q$, as in the ciphertext ring of Ring-LWE-based schemes.

Regev-style Encoding. Here we consider the input space of the encoding (the ring R over which the QRP is defined) as the ones used in HE schemes based on standard LWE, i.e. the ring $\mathbb{Z}_q$, for $q \in \mathbb{N}$. One example of such schemes is [BV11a]. Note that $\mathbb{Z}_q$ is not a field, since q is not required to be a prime. A popular choice for q is a product of co-prime numbers $q = \prod_i q_i$ with some extra conditions on q_i's as discussed in works as [Reg05, Pei09].

The encoding $\mathsf{E.Regev}$ we consider over the ring $\mathbb{Z}_q$ is the same as the one used to construct lattice-based SNARGs and SNARKs in [GMNO18, Nit19], a slight variation of the classical LWE cryptosystem initially presented by Regev [Reg05]. The encryption scheme is described by parameters $\Gamma \leftarrow (q, Q, n, \alpha)$, with $q, Q, n \in \mathbb{N}$ such that $(q, Q) = 1$, and $0 < \alpha < 1$. We will also consider $\chi_\sigma(S)$, the discrete Gaussian distribution over a discrete set S with mean 0 and parameter σ.

$\mathsf{Gen}(1^\kappa, \Gamma)$**:** Choose some random string $\boldsymbol{s} \leftarrow \mathbb{Z}_Q^n$. Output $\mathsf{sk} = \boldsymbol{s}$.

$\mathsf{E}_{\mathsf{sk}}(m)$**:** Given $m \in \mathbb{Z}_q$, sample $\boldsymbol{a} \leftarrow \mathbb{Z}_Q^n$, define $\sigma = Q\alpha$; $e \leftarrow \chi_\sigma(\mathbb{Z}_q)$. Output $C = (-\boldsymbol{a},\ \boldsymbol{a} \cdot \boldsymbol{s} + qe + m)$.

$\mathsf{D}_{\mathsf{sk}}(C)$**:** Parse $\mathsf{sk} = \boldsymbol{s}, C = (\boldsymbol{a}_0, c_1)$ Compute $m = (\boldsymbol{a}_0 \cdot \boldsymbol{s} + c_1) \mod q$.

On the suitability of the encoding. It is easy to see that this is a *statistically-correct* encoding scheme. When encodings are added together and multiplied by scalars, the noise starts to build up. Nevertheless, for any fixed ℓ there is a choice of parameters Γ such that the encoding is ℓ-*linearly-homomorphic*. Consequently, in order to ensure that we obtain a valid encoding of the expected result, we need to start with sufficiently small noise in each of the initial encodings. A more detailed discussion about the noise growth and the choices for Γ can be found in the work of Banaszczyk [Ban95] and in more recent articles [GMNO18, BISW17, BISW18] that specifically address SNARK applications of this encoding. The *quadratic root detection* and *image verification* can be implemented using D_{sk}.

Security: Regarding security, this encoding scheme was already used and conjectured as linear-only and secure against generalized q-PDH assumption over fields by prior works. Our generalized q-PDH assumption over rings extends this, taking into account encodings over rings. The constructions in [BISW17, BISW18] also employ $\mathsf{E.Regev}$ to instantiate their SNARG(K) and this is assumed to be linear-only extractable, a stronger assumption than *secure encoding*, i.e. it implicitly satisfies both the Generalized q-PDH and the Generalized Augmented q-PKE assumptions as shown in Appendix B.

Extension to Ring-LWE. Our Regev-style encoding can be extended to an encoding over the ring $\mathcal{R}_q$ used in Ring-LWE based schemes by combining N copies of the encoding above, similar to what we did in Section 6.1.

Torus Encoding. We will use a variant of the Torus FHE (TFHE) cryptosystem from [CGGI20]. We let $\mathcal{R}_\mathbb{R} = \mathbb{R}[Y]/(f(Y))$, $\mathcal{R}_\mathbb{Z} = \mathbb{Z}[Y]/(f(Y))$ and $\mathcal{R}_q = \mathbb{Z}_q[Y]/(f(Y))$ denote the quotient rings with respect to some polynomial $f(Y) = Y^N + 1$, where m is an integer and N is a power of 2. We let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the torus, which is a $\mathbb{Z}$-module structure but not a ring.

We consider the $\mathcal{R}_\mathbb{Z}$-module $\mathbb{T}_\mathcal{R} = \mathcal{R}_\mathbb{R}/\mathcal{R}_\mathbb{Z}$. The plaintext for the TFHE cryptosystem is the $\mathbb{Z}$-module $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Our encoding scheme $\mathsf{E.Torus}$ has $\mathbb{Z}_q$ as message space and will be used for encoding of elements in $\mathcal{R}_q = \mathbb{Z}_q[Y]/(f(Y))$. The key remark is that the ring $\mathcal{R}_q$ can be identified with a subgroup of the torus $\mathbb{T}^N$ via the map $\mathcal{R}_q \simeq \mathbb{Z}_q^N$ that identifies $q^{-1}\mathbb{Z}/\mathbb{Z} \simeq \mathbb{Z}_q$ as an isomorphism of $\mathbb{Z}$-modules. Also, $\mathbb{T}^N \simeq \mathbb{T}_\mathcal{R}$ because $\mathbb{T}^N$ can be seen as a vector of coefficients. The module structure of the encoding space $\mathbb{T}^{N+1}$ allows us to conjecture that $\mathsf{E.Torus}$ scheme only supports linear homomorphic operations.

Let $\mathbb{B} = \{0, 1\}$. The encoding scheme $\mathsf{E.Torus}$ is described by parameters $\Gamma \leftarrow (q, N, \alpha)$, with $q, N \in \mathbb{N}$ such that $0 < \alpha < 1$. The noise parameter α is the standard deviation for a concentrated distribution on the torus (more details can be found in [CGGI20]). Below, we describe the algorithms of the encoding:

$\mathsf{Gen}(1^\kappa, \Gamma)$**:** Choose a random vector $\boldsymbol{s} \in \mathbb{B}^N$. Output $\mathsf{sk} = \boldsymbol{s}$.

$\mathsf{E_{sk}}(m)$**:** Given $\mathsf{sk} = \boldsymbol{s} \in \mathbb{B}^N$ and $m \in \mathbb{Z}_q$, apply the map $\mathbb{Z}_q \simeq q^{-1}\mathbb{Z}/\mathbb{Z}$ to m and get $m' \in \mathbb{T}$ such that $m' \equiv m/q \mod 1$, sample a vector $\boldsymbol{a} \in \mathbb{T}^N$ and compute $b = \boldsymbol{s} \cdot \boldsymbol{a} + m' + e$ where $e \in \mathbb{T}$ is sampled according to a noise distribution defined by the standard deviation α.

$\mathsf{D_{sk}}(C)$**:** Parse $\mathsf{sk} = \boldsymbol{s}, C = (\boldsymbol{a}, b)$. Compute $m" = b - \boldsymbol{a} \cdot \boldsymbol{s} = m" + e$. Round $m"$ to the nearest point m' on the torus with respect to a distance function and apply the equivalence $q^{-1}\mathbb{Z}/\mathbb{Z} \simeq \mathbb{Z}_q$ to recover m.

On the suitability of the encoding. It is easy to see that this is a *statistically-correct* encoding scheme and due to the linearly-homomorphic property of the cryptosystem (see Appendix E for specific details), for a fixed ℓ, there is a choice of parameters Γ such that we have *ℓ-linearly-homomorphic*. The *quadratic root detection* and *image verification* can be implemented using $\mathsf{D_{sk}}$.
Security: $\mathsf{E.Torus}$ is semantically secure under the assumption TLWE, a generalized intractability problem similar to LWE. Also, it is plausible that $\mathsf{E.Torus}$ scheme only permits linear homomorphisms, therefore we conjecture that this is a *secure encoding*, satisfying both q-PDH and q-PKE assumptions. A heuristic argument for believing multiplication of two encoded values is impossible is the torus structure of the encoding space, $\mathbb{T}$ is a $\mathbb{Z}$-module and not a ring (i.e., the product of elements in $\mathbb{T}$ is not well defined), so there is no way for one to compute any missing $\mathsf{E}(s^{q+1})$ to solve q-PDH. Of course, the original encryption scheme TFHE as defined in [CGGI20] is more elaborate and overcomes this limitation: it consists of three major encryption/decryption schemes, each represented by a different plaintext space and makes use of tools like key-switching, gate bootstrapping and gadget decomposition function to perform computations other than additions. These operations are possible only if some extra keys are available, for example some precomputed ciphertexts of the binary secret key in the case of gate bootstrapping. Since we do not consider all these extensions and we do not provide encodings of the secret key in the crs, our encoding $\mathsf{E.Torus}$ is limited to basic linear operations.

7.3 (zk-)SNARKs for Ring-LWE-based homomorphic encryption

We now have everything we need to instantiate the protocol defined in Section 5.1. We pick the ring $\mathcal{R}_q = \mathbb{Z}_q[Y]/(f(Y))$, to match the ciphertext space of the Ring-LWE scheme from Section 7.1. Depending on the choice of q and $f(Y)$ in the underlying schemes, we have different options for our exceptional sets. Generally speaking, if $q = \prod_{i=1}^{k} p_i$, where $p < p_1 < p_2 < \ldots < p_k$ and p comes from the plaintext space $\mathcal{R}_p$, we can always find the exceptional set $A^* = \{1, 2, \ldots, p_1 - 1\} \subset R^*$. Hence, if p_1 is big enough we don't need to worry about anything else. Otherwise, we can move to an extension of the ciphertext ring or apply the parallel soundness amplification strategy, similar to what we did in Section 6.

We can next choose a secure encoding scheme E from the ones in Section 7.2. Assuming that the evaluation algorithm of the underlying homomorphic encryption scheme (e.g. [FV12]) does not involve modulus switching and rounding operations, we directly obtain a Designated Verifier SNARK for computation on encrypted data by invoking Theorem 6 for $\mathcal{R}_q$, as explained in Section 7.3. We could choose schemes such as BGV [BGV12] that employ modulus switching techniques, and can deal with the quadratic growth of the noise after a multiplication. In order to represent this operation as part of the arithmetic circuit represented by the QRP, we introduce a "mod q_i" gate in Appendix D. Even though the overall circuit remains over $\mathcal{R}_q$, we would like to note that there is no need to repeatedly apply the "mod q_i" gate after e.g. every addition until switching to the next modulus q_{i-1} happens. The reason behind this is that adding m elements smaller than q_i results in a value smaller than $m \cdot q_i$, thus not compromising correctness. Therefore, we only need to place this gate in the circuit whenever correctness would otherwise be lost.

Context hiding. Another challenge for our VC scheme would be preserving privacy of the inputs against the verifier. Such a property would turn useful in the following two example scenarios. In the first one, the person holding the secret key for HE and the verifier (who holds the secret key for the encoding) checking the computation over the ciphertexts are different entities. In the second scenario, the Prover wants to compute on ciphertext from the verifier using some secret coefficients (e.g. a Machine Learning model, or his own input in a two-party computation scenario) that he wants to remain private.

The *context hiding* property roughly says that output encodings together with input verification tokens do not reveal any information on the input. Note that this is required to hold even against a party that is in possession of the secret key for the encryption scheme. A formal definition of context-hiding is given in Appendix A.1. We can make our VC scheme context-hiding using the same techniques as proposed in [FNP20]. In the HE schemes we propose, information about the underlying plaintexts may be inferred from the distribution of the noise recovered during decryption of the result. To address this, the strategy is to statistically hide the noise. In a nutshell, the trick is to add to the public key some honestly generated encryptions of 0 and then ask the untrusted party to add these to the result of the computation.

Comparison with [FNP20]. The advantage of choosing our SNARK for ring computation as a candidate for the VC scheme is that the resulting scheme enables a set of optimisations on the underlying homomorphic encryption scheme leading to a total computational overhead smaller than in prior works. One main reason is that the ciphertext spaces from existing HE schemes are rings, so a QRP can be defined directly for the evaluation circuit of the HE scheme. Our SNARK, instantiated with encoding schemes that work directly over the ciphertext space, avoids the limitation of previous SNARKs for computation over fields.

The work by Fiore et al. [FNP20] relies on bilinear-group based primitives such as commitments and SNARKs, and therefore imposes specific parameters to the ciphertext space, the polynomial ring $\mathcal{R}_q = \mathbb{Z}_q[Y]/(f(Y))$, which are not optimal for the relevant homomorphic schemes known today. Moreover, they do not support modulo switching or other scaling operations. Another drawback of this work comes from the trick of moving from ciphertexts in $\mathcal{R}_q = \mathbb{Z}_q[Y]/(f(Y))$ to scalars in $\mathbb{F}_q$. This requires expensive computations on large degree polynomials in $\mathbb{Z}_q[Y]$. The prover needs to carry all the circuit computations on the ciphertext polynomials without reduction modulo $f(Y)$ along the way (where $f(Y)$ is the quotient polynomial that defines $\mathcal{R}_q = \mathbb{Z}_q[Y]/(f(Y))$). Even if this is not counted in the cost of proof generation, it is an overhead for the worker performing the homomorphic evaluation of the HE scheme. In our work, such an overhead is not necessary, our techniques allow for the worker/prover to use the existing HE schemes with their latest optimisations for computations over ciphertexts. After the HE evaluation, the prover can use the intermediate ciphertexts from the homomorphic evaluation of the circuit as witness to our SNARK. We remark that these are all elements in the ring $\mathcal{R}_q$ as opposed to large degree integer polynomials in $\mathbb{Z}_q[Y]$ computed in Fiore et al. [FNP20].

Another major advantage of our SNARK is that it supports generic rings $\mathcal{R}_q$ with $q = \prod_{i=0}^{L} q_i$ for a chain of moduli $\{q_0, \ldots, q_L\}$ as in the state-of-art leveled HE schemes. Our SNARK also enables noise reduction operations as modulo switching in the evaluation circuit to be proven. A circuit for the modulo switching procedure and its overhead in terms of QRP is detailed in Appendix D.2. A qualitative difference is that the scheme of [FNP20] is a commit-and-prove scheme; and has the inherent drawback that it is limited by the choice of schemes which are compatible with both the commitment scheme and the proof system. Our scheme is an instantiation of a SNARK without combining two different proof systems. We believe one could turn our scheme into a commit-and-prove SNARK along the lines of [AGM18] by "extracting" a suitable encoding to act as a commitment to the input wire values from the SNARK. We leave working out the details to obtain a concrete commit-and-prove scheme for ring computation to future work.

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A More Preliminaries

A.1 Verifiable Computation

Verifiable computation [GKR08, GGP10] addresses the setting where a computationally limited client wishes to outsource the computation of a function to an untrusted, but computationally powerful worker. The goal is to enable to client to outsource the computation and be able to verify the correctness of the result such that this verification is less work than the evaluation of the function itself.

Definition 7 (Verifiable Computation). *A verifiable computation scheme is a tuple of polynomial time algorithms* $(\mathsf{KGen}, \mathsf{ProbGen}, \mathsf{Compute}, \mathsf{Ver})$ *defined as follows.*

- $(\mathsf{SK}, \mathsf{PK}) \leftarrow \mathsf{KGen}(1^\kappa, F)$*: A randomized key generation algorithm takes a function* F *as input and outputs a secret key* SK*, a public key* PK_F*, and evaluation key* EK_F*.*
- $([x], \mathsf{VK}_x) \leftarrow \mathsf{ProbGen}_{\mathsf{PK}}(x)$ *A randomized problem generation algorithm takes the public key* PK_F*, an input* x*, and outputs an encoding of* x*, together with a private verification key* VK_x*.*
- $[y] \leftarrow \mathsf{Compute}_{\mathsf{PK}}([x])$ *A deterministic worker computation algorithm takes the evaluation key* EK_F*, the encoded input* $[x]$ *to compute a value* $[y]$*.*
- $y \leftarrow \mathsf{Ver}_{\mathsf{SK}}(\mathsf{VK}_x, [y])$ *A verification algorithm uses the verification key* VK_x*, the worker's output* $[y]$*, and outputs* $y \in \{0,1\}^* \cup \perp$*, where* y *is the output of the computation and* $\perp$ *indicates that the client rejects the worker's output.*

  A verifiable computation scheme satisfies correctness, efficiency and security properties.

- *Correctness. Correctness guarantees that if the worker is honest, the verification test will pass. That is, for all* F*, and for all* x *in the domain of* F*,*

$$\Pr\left(y = F(x) \;:\; \begin{array}{c} (\mathsf{SK}, \mathsf{PK}) \leftarrow \mathsf{KGen}(1^\kappa, F) \\ ([y], \mathsf{VK}_x) \leftarrow \mathsf{ProbGen}_{\mathsf{PK}}([x]) \\ [y] \leftarrow \mathsf{Compute}_{\mathsf{PK}}([x]) \\ y \leftarrow \mathsf{Ver}_{\mathsf{SK}}(\mathsf{VK}_x, [y]) \end{array} \right) = 1$$

- *Efficiency. The efficiency requirement states that the complexity of the outsourcing algorithm* $\mathsf{ProbGen}$*, and verification algorithm* Ver *together is less than the computation required to evaluate* F*. A VC myust satisfy the property that for any* x *and any* $[y]$*, the time required for* $\mathsf{ProbGen}(x)$ *plus the time required for* $\mathsf{Ver}(\mathsf{VK}_x, [y])$ *is* $o(T)$*, where* T *is the time required to compute* $F(x)$*.*
- *Security. A VC scheme is secure if a malicious worker cannot make the verification algorithm accept an incorrect answer. That is, a scheme is secure if the advantage of any PPT adversary* $\mathcal{A}$ *in the game* $\mathsf{Expt}_{\mathcal{A}}^{Ver}$ *defined as* $\Pr\left(\mathsf{Expt}_{\mathcal{A}}^{Ver}[VC, F, \kappa] = 1\right)$ *is negligible.*

Context-Hiding. An additional property that can be defined for a VC scheme is called context-hiding. This captures the setting where one wants to hide information on the input x even from the verifier. Such a property would turn useful in scenarios where the data encoder and the verifier are different entities. Informally, this property says that output encodings $[y]$, as well as the input verification tokens verification key VK_x do not reveal any information on the input x. Notably this should hold even against the holders of the secret key SK. We formalize this definition in zero-knowledge style, requiring the existence of a simulator algorithm that, without knowing the input, should generate $(\mathsf{VK}_x, [y])$ that look like the real ones. More formally:

procedure $\underline{\text{GAME } \mathsf{Expt}_{\mathcal{A}}^{Ver}(VC, F, \kappa)}$
  $(\mathsf{SK}, \mathsf{PK}) \leftarrow \mathsf{KGen}(1^\kappa, F)$
  for $i = 1, \dots, \ell = \mathsf{poly}(\kappa)$ **do**
    $x_i = \mathcal{A}(\mathsf{PK}, x_1, [x_1], \dots, x_{i-1}, [x_{i-1}])$
    $([x_i], \mathsf{VK}_{x_i}) \leftarrow \mathsf{ProbGen}_{\mathsf{PK}}(x_i)$
  end for
  $(i, [y]) = \mathcal{A}(\mathsf{PK}, x_1, [x_1], \dots, x_\ell, [x_\ell])$
  $y \leftarrow \mathsf{Ver}_{\mathsf{SK}}(\mathsf{VK}_{x_i}, [y])$
return $((y \neq \perp) \wedge (y \neq F(x_i)))$
end procedure

Definition 8 (Context-Hiding). *A VC scheme is context-hiding for a function F if there exist simulator algorithms S_1, S_2 such that:*

- *the keys* $(\mathsf{SK}, \mathsf{PK})$ *and* $(\mathsf{SK}', \mathsf{PK}')$ *are statistically indistinguishable, where* $(\mathsf{SK}, \mathsf{PK}) \leftarrow \mathsf{KGen}(1^\kappa, F)$ *and* $(\mathsf{SK}', \mathsf{PK}', \mathsf{td}) \leftarrow S_1(1^\kappa, f)$*;*
- *for any input x, the following distributions are negligibly close*

$$(\mathsf{SK}', \mathsf{PK}', \mathsf{VK}_x, [x], [y]) \approx (\mathsf{SK}', \mathsf{PK}', \mathsf{VK}'_x, [x], [y]')$$

  where $(\mathsf{SK}', \mathsf{PK}', \mathsf{td}) \leftarrow S_1(1^\kappa, f)$, $([x], \mathsf{VK}_x) \leftarrow \mathsf{ProbGen}_{\mathsf{PK}'}(x)$,
  $[y] \leftarrow \mathsf{Compute}_{\mathsf{PK}}([x])$, *and* $([y]', \mathsf{VK}'_x) \leftarrow S_2(\mathsf{td}, \mathsf{SK}', F(x))$.

B Assumptions on Ring Encodings

Consider an encryption scheme which satisfies the properties required for an encoding scheme (see Definition 6). If the encryption scheme can be assumed to be linear-only extractable, which is the assumption in [BCI+13, BISW17, BISW18], then it automatically is a *secure* encoding, i.e. it satisfies both the Generalized q-PDH and the Generalized Augmented q-PKE assumptions. We recall the linear-only extractable definition from [BCI+13], which we adapt to the broader context of (non-field) commutative rings with identity.

Definition 9 (Linear-only extractable). *An encoding scheme* $\mathsf{Encode} = (\mathsf{Gen}, \mathsf{E})$ *over R is linear-only extractable if for all probabilistic polynomial time algorithms $\mathcal{A}$, there exists a probabilistic polynomial time extractor $\chi_{\mathcal{A}}$ such that the following probability is negligible in the security parameter.*

$$\Pr\left(c \neq a_0 + \sum_{i=1}^n a_i x_i \;:\; \begin{array}{c} (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^\kappa), \\ x_1, \dots, x_n \xleftarrow{R} R, \\ \sigma = (\mathsf{pk}, \mathsf{E}(x_1), \dots, \mathsf{E}(x_n)), \\ (\mathsf{E}(c); a_0, \dots, a_n) \leftarrow (\mathcal{A}||\chi_{\mathcal{A}})(\sigma) \end{array} \right).$$

Lemma 6. *If an encoding scheme* $\mathsf{Encode} = (\mathsf{Gen}, \mathsf{E})$ *is IND-CPA secure and linear-only extractable, then it is an encoding scheme that satisfies Generalized Augmented q-PKE (Assumption 2).*

Proof. Let $\sigma = (\mathsf{pk}, \mathsf{E}(1), \mathsf{E}(s), \dots, \mathsf{E}(s^q), \mathsf{E}(\alpha), \mathsf{E}(\alpha s), \dots, \mathsf{E}(\alpha s^q))$. We will show that Encode satisfies q-PKE, meaning we will show that for any adversary $\mathcal{A}$ able to produce $c, \hat{c}$ such that $\alpha c - \hat{c} = 0$, there exists an extractor $\chi_{\mathcal{A}}$ which outputs coefficients a_i satisfying $c = \sum_{i=0}^q a_i s^i$ with non negligible probability.

We define two adversaries $\mathcal{B}_c$ and $\mathcal{B}_{\hat{c}}$ that, upon receiving as input σ, run exactly the same code as $\mathcal{A}$ and output, respectively, c and $\hat{c}$. By our linear-only extractable assumption on E, there exist an extractor χ_c (resp. $\chi_{\hat{c}}$) for $\mathcal{B}_c$ (resp. $\mathcal{B}_{\hat{c}}$) which outputs $a_0, \ldots, a_q, b_0, \ldots, b_q$ (resp. $a'_0, \ldots, a'_q, b'_0, \ldots, b'_q$) such that

$$c = \sum_{i=0}^{q} a_i s^i + \sum_{i=0}^{q} b_i \alpha s^i, \quad \hat{c} = \sum_{i=0}^{q} a'_i s^i + \sum_{i=0}^{q} b'_i \alpha s^i$$

with non negligible probability.

Knowing that $\alpha c - \hat{c} = 0$ implies either that the polynomial

$$P(X, Y) = X^2 \sum_{i=0}^{q} b_i Y^i + X \sum_{i=0}^{q} (a_i - b'_i) Y^i - \sum_{i=0}^{q} a'_i Y^i$$

is the zero polynomial, or that (α, s) are roots of $P(X, Y)$. We rule out the second case by the IND-CPA security of the encoding scheme and the generalized Schwartz-Zippel lemma. Hence, $P(X, Y) = 0$, which gives us that for every $i \in [q]$, $b_i = a'_i = 0$ and $a_i = b'_i$. Therefore, we have defined an extractor $\chi_{\mathcal{A}}$ for the Generalized Augmented q-PKE assumption, which outputs the coefficients a_i obtained from χ_c. ■

Lemma 7. *If an encoding scheme* $\mathsf{Encode} = (\mathsf{Gen}, \mathsf{E})$ *is IND-CPA secure and linear-only extractable, then it is an encoding scheme that satisfies the Generalized q-PDH assumption (Assumption 1).*

Proof. Consider an adversary $\mathcal{A}$ that breaks q-PDH of the scheme Encode. We construct an adversary $\mathcal{B}$ that breaks IND-CPA. Consider the adversary $\mathcal{B}$ playing left-or-right oracle game where the adversary gets access to an encryption oracle that receives a pair of chosen messages always returns a ciphertext encrypting either the left or the right message. The adversary wins if it guesses the left-or-right bit.

$\mathcal{B}$ samples s_0, s_1 uniformly from an exceptional set $A^* \subset R^*$. $\mathcal{B}$ gets access to the left-or-right encryption oracle, makes queries on pairs (s_0^k, s_1^k) for $k \in \{0, \ldots, q, q+2, \ldots, 2q\}$, and receives $\{\mathsf{E}(s_b^i)\}_{i=0}^{2q, i \neq q+1}$ for challenge bit b. $\mathcal{B}$ now runs the q-PDH adversary $\mathcal{A}$ on $\{\mathsf{E}(s_b^i)\}$. $\mathcal{A}$ returns $y \in \{\mathsf{E}(s_b^{q+1})\}$. $\mathcal{B}$ now invokes the extractor that exists since Encode satisfies linear-only extractability (c.f. Definition 9). $\chi_{\mathcal{A}}$, given the same input as $\mathcal{A}$ and its internal randomness, returns $a_0, \cdots, a_q, a_{q+2}, a_{2q}$ such that $a_0 + \sum_{i=1}^{2q, i \neq q+1} a_i s_b^i = s_b^{q+1}$. Since $\mathcal{B}$ knows s_0, s_1, it checks whether $a_0 + \sum_{i=1}^{2q, i \neq q+1} a_i s_0^i = s_0^{q+1}$ or $a_0 + \sum_{i=1}^{2q, i \neq q+1} a_i s_1^i = s_1^{q+1}$, and outputs the bit b^* for which this holds. Notice that the previous strategy will output a *single* possible value for b^* with high probability, which further matches the challenge bit b. This is because, for the random s_{1-b}, we have that $a_0 + \sum_{i=1}^{2q, i \neq q+1} a_i s_{1-b}^i = s_{1-b}^{q+1}$ will hold only with probability $q/|A^*|$, by the generalized Schwartz-Zippel lemma. ■

Informally, the linear-only extractability assumption captures the fact that an adversary can perform only affine operations over the encodings provided as input. It can be argued that the PDH asssumption is in some sense weaker than linear-only extractability since the former is implied by the latter. However, if for an encoding scheme like JL, the linear-only extractability property is broken, computing non-linear homomorphisms would be possible which would mean efficient fully homomorphic encryption which is not known using current techniques. In [CRFG19], the authors consider a seemingly related notion called enhanced CPA and show that an additively homomorphic encryption scheme over $\mathbb{Z}_{2^k}$ cannot satisfy enhanced CPA. We note that their attack relies on the fact that the adversary has access to an oracle that checks the validity of a ciphertext. In our use of a encoding scheme in constructing a SNARK, we are concerned only with one-time soundness and our setting does not provide access to such oracles to the adversary (see also the remark at the end of Section 2). In proving multi-theorem soundness of designated-verifier SNARK constructions, one needs to make a stronger assumption called the q power-knowledge of equality (q-PKEQ) assumption. The following q-PKEQ assumption is needed in the designated verifier setting where the adversary has access to a verification oracle (in the public verification setting, this is for free and the adversary has no additional advantage). This assumption is invoked to prove multi-statement soundness in the proof to test if two (potentially adversarially generated) encodings have the same value underneath without having the secret key.

Assumption 3 (Generalized q-PKEQ) *The generalized q power-knowledge of equality assumption holds for an encoding scheme* Encode *if for all non-uniform probabilistic polynomial time algorithm $\mathcal{A}$, there exists a non-uniform probabilistic polynomial time extractor $\chi_{\mathcal{A}}$ such that the following probability is negligible in the security parameter.*

$$\Pr\left(\begin{array}{c}(b = 0 \wedge \hat{c} \in \{\mathsf{E}(c)\})\\ \vee\\ (b = 1 \wedge \hat{c} \notin \{\mathsf{E}(c)\})\end{array} : \begin{array}{c}(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^{\kappa}),\\ s \xleftarrow{R} A^*,\\ \sigma = (\mathsf{pk}, \mathsf{E}(1), \mathsf{E}(s), \ldots, \mathsf{E}(s^q), \mathsf{E}(s^{q+2}), \ldots, \mathsf{E}(s^{2q})),\\ (\mathsf{E}(c), \hat{c}; b) \leftarrow (\mathcal{A}||\chi_{\mathcal{A}})(\sigma)\end{array}\right).$$

C QRP as an Abstraction

In this section, we highlight the generality of our notion of QRP and our construction by outlining how our notion recovers the QPP based construction of [KPP+14] for polynomial circuits. We sketch how the SNARK construction via QPP is a special case of our construction via QRP below.

QPP as an instantiation of QRP. The following definition is recovered by Definition 5, where $R = \mathbb{F}_p[Z], A = \mathbb{F}_p \subset R$, i.e. the degree-zero polynomials, and $A^* = \mathbb{F}_p^*$. The bivariate polynomial $p(x, z)$ accounts for the wire values themselves being polynomials.

Definition C1 (Quadratic Polynomial Program (QPP) [KPP+14]) *A QPP Q consists of three sets of polynomials, $\mathcal{V} = \{v_k(x)\}, \mathcal{W} = \{w_k(x)\}, \mathcal{Y} = \{y_k(x)\}$ and a target polynomial $t(x)$. Let C be a polynomial circuit. We say that Q computes C if the following holds:*

$a_1(z), \ldots, a_n(z), a_{m-n'+1}(z), \ldots a_m(z)$ is a valid assignment to the input/output variables of C if and only if there exist polynomials $a_{n+1}(z), \ldots, a_{m-n'}(z)$ such that $t(x)$ divides $p(x, z)$, where

$$p(x, z) = \left(\sum_{k=1}^{m} a_k(z) \cdot v_k(x)\right) \cdot \left(\sum_{k=1}^{m} a_k(z) \cdot w_k(x)\right) - \left(\sum_{k=1}^{m} a_k(z) \cdot y_k(x)\right)$$

The degree of Q is said to be $\deg(t(x))$.

D Some useful QRPs

While the QRP construction described in Section 3 would allow us to easily describe arithmetic circuits over e.g. $\mathbb{Z}_{2^k}$ or the rings R_q used for homomorphic encryption, in practical scenarios one is also interested in performing bit-wise operations such as comparisons and, as it is specially the case in some levelled HE schemes, modular reduction.

D.1 Bit Decomposition Gate

We show how to build a QRP which, given an input $a \in R$, gives as an output wires holding values $a_i \in \{0,1\}$ which correspond to the 'binary representation' of a. Our following description is specialized for $R = GR(2^k, d)$, but it can be easily adapted to other rings such as those employed in Section 7.1.

We provide two different versions of this gate. For the first one, nothing is known about a, whereas in the second case, better efficiency is achieved by assuming that $a \in \mathbb{Z}_{2^k}$. When interested in computation over $\mathbb{Z}_{2^k}$ only, the former version of the gate where potentially $a \notin \mathbb{Z}_{2^k}$ is necessary only if the prover is providing some inputs to the QRP in a zero-knowledge way. Nevertheless, once the inputs from the prover have been asserted to be elements of $\mathbb{Z}_{2^k}$, one can use the more efficient $\mathbb{Z}_{2^k}$-splitter gate during the rest of the circuit. The provers inputs can be tested to be from $\mathbb{Z}_{2^k}$ either by inspection when those are provided in the clear, or when they are provided in ZK, by e.g. applying the general R-splitter gate to them and outputting to the verifier all the wires that should be always equal to zero in a 'binary representation' of an element in $\mathbb{Z}_{2^k} \subset R$. Let $A \subset R$ be the exceptional set.

1. $\mathbb{Z}_{2^k}$-splitter gate: This mini-QRP has one input wire, holding $a \in \mathbb{Z}_{2^k}$, and k output wires holding $a_1, \ldots, a_k \in \{0,1\}$ such that $a = \sum_{i=1}^{k} 2^{i-1} a_i$. Label the input wires as $1, \ldots, k$ and the output wire as $k+1$. Let $t(x) = (x - r) \prod_{i=1}^{k} (x - r_i)$, where $r, r_1, \ldots, r_k \in A$ are pairwise different. In an approach similiar to Pinocchio [PHGR13], we set:

$$\begin{aligned} v_0(r) = 0, v_i(r) = 2^{i-1}, \text{ for } 1 \le i \le k, v_{k+1}(r) = 0, \\ w_0(r) = 1, w_i(r) = 0, \text{ for } 1 \le i \le k, w_{k+1}(r) = 0, \\ y_0(r) = 0, y_i(r) = 0, \text{ for } 1 \le i \le k, y_{k+1}(r) = 1 \end{aligned}$$

   For $1 \le j \le k$:

$$\begin{aligned} v_j(r_j) = 1, v_i(r_j) = 0 \text{ for all } i \ne j, \\ w_0(r_j) = 1, w_j(r_j) = -1, w_i(r_j) = 0 \text{ for all } i \ne 0, j, \\ y_i(r_j) = 0 \text{ for all } i \end{aligned}$$

   If $(v_0(x) + \sum a_k v_k(x)) \cdot (w_0(x) + \sum a_k w_k(x)) - (y_0(x) + \sum a_k y_k(x))$ is divisible by $t(x)$, then it must be 0 at r, and therefore, by the first set of equations, this gives, $a = \sum_{i=1}^{k} 2^{i-1} a_i$. The second set of equations guarantee that each r_j is a root, which implies, $a_j(1 - a_j) = 0$. Since all the zero divisors of R belong to the maximal ideal (2), it follows that if a_j is a zero divisor then $a_j \pm 1$ is not, and thence the only solutions for the previous equation are $a_j \in \{0,1\}$. Together, these give the guarantee that all a_i are bits, and are the binary decomposition of a.
2. R-splitter gate: This works essentially as the previous version of the splitter gate repeated δ times in parallel, once for every component of R seen as a free-module of rank δ over $\mathbb{Z}_{2^k}$.

D.2 Modular reduction gate

In leveled homomorphic schemes, namely capable of evaluating circuits of arbitrary size, but known beforehand, without involving the costly bootstrapping procedure, the key tool is the modulus switching procedure which allows to switch a ciphertext encrypted under a modulus q to a smaller modulus q_0 in order to keep the noise level "constant". Hence by selecting a chain of moduli $\{q_0, \ldots, q_L\}$ long enough to perform the desired computations, bootstrapping is no longer needed. The modulus switching allows to decrease the size of the noise of a level $j > 0$ ciphertext, as soon as it becomes too important. Roughly, the idea is to drop one (or several) levels in the ladder of moduli by scaling the ciphertext by q_i/q_j for $i < j$, which roughly scales down the noise by the same factor.

Of special interest for the application of SNARKs over Homomorphic Encryption schemes is the fact of having a way to compute modular reductions at a reasonable cost. We provide below a *"mod q_i"* gate, which has a cost of $(\lceil \log q \rceil + \lceil \log q_i \rceil + 3) \cdot d$ multiplication gates in the underlying QRP over R_q, where $R_q = \mathbb{Z}_q[Y]/(f(Y))$ and $d = deg(f(Y))$. The cost of the gate can be further optimized to $(\lceil \log q \rceil + \lceil \log q_i \rceil + 1) \cdot d + 2$, as we will sketch after its more expensive but simpler to explain implementation.

Let $Q = \lceil \log q_i \rceil$. What our QRP will prove is the following: Given $z \in R_q$, expressed as $z = \sum_{\ell=0}^{d-1} z_\ell \cdot Y^\ell, z_\ell \in \mathbb{Z}_q$, the Prover can prove that $\tilde{z} = \sum_{\ell=0}^{d-1} \tilde{z}_\ell \cdot Y^\ell$, where $\tilde{z}_\ell \equiv z_\ell \mod q_i$. He does so by providing, for each $\ell \in [d]$, values $x_\ell, \tilde{z}_\ell, t_\ell \in \mathbb{Z}_q$ (where the two latter are actually provided in a bit-decomposed manner) such that:

$$\tilde{z}_\ell = \begin{cases} \sum_{k=0}^{Q-1} 2^k \cdot \tilde{z}_{\ell,k}; & \tilde{z}_{\ell,k} \in \{0,1\} \\ z_\ell - x_\ell \cdot q_i \end{cases} \tag{9}$$

$$t_\ell = \begin{cases} \sum_{k=0}^{\lceil \log q \rceil -1} 2^k \cdot t_{\ell,k}; & t_{\ell,k} \in \{0,1\} \\ \tilde{z}_\ell - q_i \end{cases} \tag{10}$$

$$f_{z_\ell} = \sum_{k=Q}^{\lceil \log q \rceil -1} t_{\ell,k} \neq 0 \tag{11}$$

Equation (9) can be verified by a slightly optimized Bit Decomposition gate, which costs $Q+1$ multiplication gates. The purpose of this part of the circuit is proving both that $\tilde{z}_\ell$ is a representative of the class $z_\ell \mod q_i$ smaller than 2^k. Note that we are not done at this point, as the Prover could be providing a value $\tilde{z}_\ell$ such that $q_i < \tilde{z}_\ell < 2^k$. This is ruled out by the combination of Equations (10) and (11), which ensure that $\tilde{z}_\ell - q_i > 2^k$. Their cost is that of a Bit Decomposition for the former (i.e. $\lceil \log q \rceil + 1$ multiplication gates) and one multiplication gate for the latter. In more detail, we check that every f_{z_ℓ} which results form computing $\tilde{z}_\ell \equiv z_\ell \mod q_i$ is not zero by checking whether $\prod_{z_\ell} f_{z_\ell} \neq 0$[6].

The more efficient implementation of the "mod q_i" gate, costing $(\lceil \log q \rceil + \lceil \log q_i \rceil + 1) \cdot d + 2$ multiplication gates, can be built in almost the same way as in our previous exposition, but "packing" each of the equality tests at the bottom part of Equations (9) and (10) into a single equality check over R_q. More specifically, these can be implemented as:

$$\tilde{z} = \sum_{\ell=0}^{d-1} \tilde{z}_\ell \cdot Y^\ell = z - q_i \cdot (\sum_{\ell=0}^{d-1} x_\ell \cdot Y^\ell) \tag{12}$$

[6] Note that f_{z_ℓ}, which is the sum of a few bits, will not be a zero divisor in our concrete application. One should be careful to deal with such case in more general scenarios.

$$t = \sum_{\ell=0}^{d-1} t_\ell \cdot Y^\ell = \tilde{z} - q_i \cdot (\sum_{\ell=0}^{d-1} Y^\ell) \tag{13}$$

E Further details on SNARKs for computation over Encrypted Data

E.1 Further details on Torus encoding

Multiplying encoded elements with elements from R: We next show explicitly how our TFHE-based encoding is R-linear homomorphic. $R = \mathbb{Z}_m[Y]/(f(Y))$ is a free module over $\mathbb{Z}_m$ of rank d, i.e. we can find a basis for R. Let ξ be a root of $f(Y)$, we have that $\{1, \xi, \ldots, \xi^{d-1}\}$ is one of such basis. The map $\phi : R \to (\mathbb{Z}_m)^d$, which sends $b = b_0 + \cdots + b_{d-1}\xi^{d-1}$ to $\phi(b) = (b_0, \ldots, b_{d-1})$ is an isomorphism of $\mathbb{Z}_m$-modules. We will make extensive use of this isomorphism going forward.

The encoding we use is the following:

$$\begin{aligned} \mathsf{E}_{\mathsf{pk}} &: R \to (\mathbb{T})^d \\ a &\mapsto \mathsf{E}_{\mathsf{pk}}(a) = (\mathsf{TFHE}(a_0), ..., \mathsf{TFHE}(a_{d-1})) \end{aligned}$$

For our QRPs, we wish to compute values of the form $E(a \cdot b)$, where $a, b \in R$, given $E(a)$ and b. The problem is that $E(a) \in (\mathbb{T})^d$, and the torus does not allow us to simply and directly compute $b \cdot E(a)$ as in previous occasions. Rather, we have to look at the R-module endomorphism $\cdot_b$ which is induced by multiplication of any element of R with b, and use this to manipulate the d individual values $\mathsf{TFHE}(a_0), ..., \mathsf{TFHE}(a_{d-1}) \in \mathbb{T}$.

In a more explicit and step-by-step fashion, $\cdot_b$ is an R-module endomorphism and hence a $\mathbb{Z}_m$-module homomorphism $\cdot_b : (\mathbb{Z}_m)^d \to (\mathbb{Z}_m)^d$. We can therefore represent this operation as follows:

$$\begin{aligned} \cdot_b &: (\mathbb{Z}_m)^d \to (\mathbb{Z}_m)^d \\ a &\mapsto M_b \cdot a \end{aligned}$$

where $M_b \in \mathcal{M}_{d\times d}(\mathbb{Z}_m)$. As a side note, in fact, M_b can be easily defined from the polynomial $f(Y)$ used to construct $R \simeq (\mathbb{Z}_m)^d$. Our goal can now be re-stated as computing $E(\cdot_b(a))$, given $E(a)$ and $b \in R$. We are almost done, as $\mathsf{TFHE}(x) + \mathsf{TFHE}(y) = \mathsf{TFHE}(x + y)$ and $\mathbb{T}$ allows for external multiplication with elements in $\mathbb{Z}$. In full formalism, let $N_b \in \mathcal{M}_{d\times d}(\mathbb{Z})$ such that $N_b \equiv M_b \mod n$. We only need to compute:

$$N_b \cdot E(a) = E(N_b \cdot a) = E(M_b \cdot a) = E(\cdot_b(a)) = E(a \cdot b)$$

F [Gro16]-Like Construction based on Linear-Only Encodings

We construct a zk-SNARK scheme for ring computations with efficiency close to its field-restricted counterpart proposed in [Gro16].

Let C be an arithmetic circuit over R, with m wires and d multiplication gates. Let $Q = (t(x), \{v_k(x), w_k(x), y_k(x)\}_{k=0}^m)$ be a QRP which computes C. We denote by $I_{io} = 1, 2, \ldots \ell$ the indices corresponding to the public input and public output values of the circuit wires and by $I_{mid} = \ell + 1, \ldots m$, the wire indices corresponding to non-input, non-output intermediate values. Let $\mathsf{Encode} = (\mathsf{Gen}, \mathsf{E})$ be a secure encoding scheme and $A^* \subset R^*$ an exceptional set.

Our scheme is based on the assumption of linear-only encodings and consists in 3 algorithms $\mathsf{RingSNARK} = (\mathsf{Setup}, \mathsf{Prove}, \mathsf{Verify})$ described in Figure 3.

$\underline{\mathsf{Setup}(1^\kappa, \mathcal{R}):}$

$\alpha, \beta, \gamma, \delta \leftarrow R^*, \quad s \leftarrow A^*, \qquad (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^\kappa)$

$\mathsf{crs} = \Big(\mathsf{pk}, \{\mathsf{E}(s^i)\}_{i=0}^{d-1}, \ \{\mathsf{E}(\frac{\beta v_k(s)+\alpha w_k(s)+y_k(s)}{\gamma})\}_{k\in I_{io}},$

$\{\mathsf{E}(\frac{\beta v_k(s)+\alpha w_k(s)+y_k(s)}{\delta})\}_{k\in I_{mid}}, \ \{\mathsf{E}(\frac{s^i t(s)}{\delta})\}_{i=0}^{d-1}\Big)$

$\mathsf{vk} = (\mathsf{sk}, \mathsf{crs}, s, \alpha, \beta, \gamma, \delta)$

return $(\mathsf{crs}, \mathsf{vk})$

$\underline{\mathsf{Prove}(\mathsf{crs}, u, w)}$

$u = (a_1, \ldots, a_\ell), \ a_0 = 1$

$w = (a_{\ell+1}, \ldots, a_m)$

$v(x) = \sum_{k=0}^m a_k v_k(x)$

$v_{mid}(x) = \sum_{k\in I_{mid}} a_k v_k(x)$

$w(x) = \sum_{k=0}^m a_k w_k(x)$

$w_{mid}(x) = \sum_{k\in I_{mid}} a_k w_k(x)$

$y(x) = \sum_{k=0}^m a_k y_k(x)$

$y_{mid}(x) = \sum_{k\in I_{mid}} a_k y_k(x)$

$h(x) = \frac{(v(x)w(x)-y(x))}{t(x)}$

$f_{mid} = \frac{\beta v_{mid}(s)+\alpha w_{mid}(s)+y_{mid}(s)}{\delta}$

$A = \mathsf{E}\big(\alpha + v(s)\big)$

$B = \mathsf{E}\big(\beta + w(s)\big)$

$C = \mathsf{E}(f_{mid} + \frac{t(s)h(s)}{\delta})$

return $\pi = (A, B, C)$

$\underline{\mathsf{Verify}(\mathsf{vk}, u, \pi)}$

$\pi = (A, B, C)$

$A = \mathsf{E}(A_v)$

$B = \mathsf{E}(B_w)$,

$C = \mathsf{E}(C_y)$

$v_{io}(x) = \sum_{i=0}^\ell a_i v_i(x)$

$w_{io}(x) = \sum_{i=0}^\ell a_i w_i(x)$

$y_{io}(x) = \sum_{i=0}^\ell a_i y_i(x)$

$f_{io} = \frac{\beta v_{io}(s)+\alpha w_{io}(s)+y_{io}(s)}{\gamma}$

$F = \mathsf{E}(f_{io})$

Check on encodings

$AB = \mathsf{E}(\alpha)\mathsf{E}(\beta) + \gamma F + \delta C$

i.e.

$A_v B_w = \alpha\beta + \gamma f_{io} + \delta C_y$

Fig. 3. RingSNARK Construction from Linear-only Encodings.

Theorem 7. *Let R be commutative ring with identity with an exceptional subset A, and d be an upper bound on the degree of the QRP. Assuming that the linear-only extractable assumption as per Definition 9 holds for the encoding scheme* Encode *over R (and A^*), the protocol* RingSNARK *described in Fig. 3 is a SNARK as per Definition 1, with soundness error $1/|A^*|$.*

F.1 Proof of Security

We first give a variant of the Schwartz-Zippel lemma for Laurent polynomials over rings that we will rely on in the proof.

Lemma 8. *Let A be an exceptional set. Let $h(X) \in R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ where no term in any X_i has degree less than $-D$ or larger than D. Let us assume that $h(X)$ is not the zero-polynomial. Let $\boldsymbol{a} \in (A)^n$ be chosen uniformly at random. Then*

$$\Pr[h(\boldsymbol{a}) = 0] \leq \frac{2nD}{|A|}.$$

Proof. We notice that $f(X) := \prod_{i=1}^n X_i^D \cdot h(X)$ is an ordinary polynomial of degree $\leq 2nD$. Since $h(\boldsymbol{a}) = 0$ implies $f(\boldsymbol{a}) = 0$, by the generalized Schwartz-Zippel lemma (Lemma 2), we have that

$$\Pr[h(\boldsymbol{a}) = 0] \leq \Pr[f(\boldsymbol{a}) = 0] \leq \frac{2nD}{|A|},$$

finishing the proof. ∎

We are now ready to give the security proof of our scheme RingSNARK:

Theorem 8. *Let R be commutative ring with identity with an exceptional subset A, and d be an upper bound on the degree of the QRP. Assuming that the linear-only extractable assumption as per Definition 9 holds for the encoding scheme* Encode *over R (and A^*), the protocol* $\mathsf{RingSNARK}$ *described in Fig. 3 is a SNARK as per Definition 1, with soundness error $1/|A^*|$.*

Proof. **Completeness.** Completeness of the SNARK protocol follows by QRP completeness and by the (statistical) correctness of the Encode scheme.
Knowledge Soundness. We will show the existence of an extrator that on same input and random coins as $\mathcal{A}$ can produce a valid witness whenever the prover $\mathcal{A}$ outputs a valid proof. Let $\mathcal{A}$ be the PPT adversary in the game for knowledge soundness (Definition 1) able to produce a proof π for which the verification algorithm returns true. By linear-only extractable assumption 9 we can run an extractor that gives us a vector of coefficients $A_\alpha, A_\beta, A_\gamma, A_\delta, \{\mathcal{A}_k\}_{k=0}^m$ and polynomials $A(x), A_h(x)$ of degree $d-1, d-2$ such that the value encoded in the proof element A can be written as a linear combination of the initial values encoded in the crs:

$$\begin{aligned} A_v = A_\alpha \alpha + A_\beta \beta + A_\gamma \gamma + A(s) + \sum_{k=0}^{\ell} A_k \frac{\beta v_k(s) + \alpha w_k(s) + y_k(s)}{\gamma} + \\ + \sum_{k=\ell+1}^{m} A_k \frac{\beta v_k(s) + \alpha w_k(s) + y_k(s)}{\delta} + A_h(s) \frac{t(s)}{\delta} \end{aligned} \tag{14}$$

We can write out B_w and C_y in a similar fashion. We can see the verification equation as an equality of multivariate Laurent polynomials. By Lemma 8, $\mathcal{A}$ has negligible success probability unless the verification equation holds when viewing A_v, B_w and C_y as formal polynomials in indeterminates $x_\alpha, x_\beta, x_\gamma, x_\delta, x_s$.

Using the verification test equations and following the same reasoning as the proof in [Gro16] we eliminate coefficient by coefficient until we obtain:

$$A(x) = \sum_{k=0}^{m} a_k v_k(x), \quad B(x) = \sum_{k=0}^{m} a_k w_k(x), \quad C(x) = \sum_{k=0}^{m} a_k y_k(x).$$

This implies that $w = (a_{\ell+1}, \ldots, a_m)$ is a witness for $u = (a_1, \ldots, a_\ell)$. ∎